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This final report summarizes the results of an evaluation of two new plate and shell elements for use in large deflection nonlinear analysis. The elements are of the "stability element" type, in which nonlinear strains are included in calculations, with their values optimized by added membrane displacement functions and special types of elemental level constraints. The report summarizes the formulation and implementation of the elements, and discusses numerical results in detail. Conclusions are drawn regarding the effectiveness of these elements for solving highly nonlinear problems and also for solving certain next page

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LEVEL II

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# INVESTIGATION OF FINITE ELEMENTS FOR STRONGLY NONLINEAR PROBLEMS

FINAL REPORT

F44620-76-C-0130

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October 31, 1978





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## ABSTRACT

Final Report - Investigation of
Finite Elements for Strongly Nonlinear Problems

This final report summarizes the results of an evaluation of two new plate and shell elements for use in large deflection nonlinear analysis. The elements are of the "stability element" type, in which nonlinear strains are included in calculations, with their values optimized by added membrane displacement functions and special types of elemental level constraints. The report summarizes the formulation and implementation of the elements, and discusses numerical results in detail. Conclusions are drawn regarding the effectiveness of these elements for solving highly nonlinear problems and also for solving certain types of linear shell structure problems. The crucial role of stepping/iterative solution procedures is discussed, and solution procedures of improved convergence are described. Finally nonlinear finite element alternatives are discussed from the points of view of accuracy, computing cost, element size, discretization considerations, and solution convergence.

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## 1.0 INTRODUCTION AND SUMMARY

This report discusses the development and evaluation of two new finite elements for nonlinear shell analysis. The elements are of a new type called "stability elements," specifically adapted to handle linear shell analysis, and nonlinear effects in plates and shells due to large deflections. The original work on these elements was under NASA-MSFC Contract NAS8-30626, monitored by Dr. John Key, and accomplished theoretical development and a partial computer code development. The present research is under AFOSR Contract F44620-76-C-0130, monitored by Mr. William Walker and Lt. Col. Joseph Morgan. Under the present contract the computer codes have been completed, critical evaluations made of the performance of the elements, and recommendations developed for further research.

The numerical performance of the stability elements has been very good, and it appears clear that they have a significant advantage over conventional elements for solving nonlinear problems without resorting to very small elements. The present research has defined modified and simplified formulations for the elements which should improve their performance still further. In addition, the numerical work has indicated that improved nonlinear solution procedures should be used in conjunction with the new elements, in order to achieve the large step sizes which these elements can handle. Recommendations are made for a combination of nonlinear, stability-type elements with nonlinear-step solution procedures.

Computing costs with the new elements have been quite high. However, high computing costs are generally acceptable in stepping solutions of highly nonlinear problems. It appears that with the use of conventional elements, in smaller sizes, to achieve comparable accuracy, the costs would be higher still. The use of the recommended nonlinear-step solution procedure, in conjunction with the stability elements, should decrease computing costs significantly through allowing larger step sizes.

This report describes the technical formulation of the stability elements and outlines the overall computational procedures required for their use. Numerical calculations are discussed for several problems in order to illustrate the elements' performance in critical areas. Solution procedure difficulties are discussed, and a special, quasi-nonlinear solution procedure which was successful in achieving convergence with relatively large step sizes is described. Emphasis in the report is placed on the quadrilateral element, since the work has indicated that it is the better of the two elements studied. There are alternative approaches to the stability elements which might be used to solve highly nonlinear problems with varying degrees of success and cost. These are briefly discussed in Section 2.6. Finally, conclusions and recommendations for further research are discussed in Section 3.0.

## 2.0 TECHNICAL DISCUSSION

This section discusses the conceptual basis, theoretical development, computer code arrangement, compatible stepping solution procedures, and numerical performance of the new stability elements. This comprises an overall description of the element development from conception through the evaluations and extensions accomplished on the present contract. This broad coverage will describe the elements sufficiently that reference to earlier documentation for detail should not be required. However, should details such as equations or more in-depth discussions be required, the reader is referred to the original contract document (Reference 1) submitted to NASA-MSFC (NAS8-30626, Volume II).

The evaluation of the numerical performance of the elements, Section 2.5, includes discussions of specific areas in which their formulation requires improvement. These results are the basis of the recommendations made in Section 3.0.

### 2.1 Description of Stability Elements

It has been found that conventional finite elements may suffer serious loss of accuracy in application to large deflection problems which are strongly nonlinear. The particular difficulty of concern here occurs when the finite element formulation specifically employs nonlinear strain-displacement equations and thus may generate large strain levels through nonlinear behavior. The problem was originally discussed by Mallett in Reference 2, by Haftka, Mallett, and Nachbar in Reference 3, and by Berke and Mallett in Reference 4. The work in Reference 3 suggested a procedure for resolving the difficulty for beam elements. The present work has extended this procedure to the two dimensional case of plates and doubly curved shells, and refers to the elements as HMN elements, in recognition of the authors of Reference 3.

The basic difficulty occurs because the nonlinear strain-displacement equations contain squares and products of the displacement gradients, in particular, the bending slopes of the midsurface of the plate or shell. Since the bending displacements of such elements conventionally are second, third, or higher degree polynomials, the squares and products of their derivatives are of degree two, three, four, or higher. The membrane strains therefore contain terms, due to nonlinear behavior, which are of a higher polynomial degree than those which result from the derivatives of the membrane displacements themselves. These higher degree polynomial forms also occur in shell elements in linear analysis, due to the presence of initial curvature. The high degree membrane strains induced by nonlinearity or initial curvature cannot be "erased" by the lower degree element membrane displacement function derivatives. The result is excessive strain energy, and hence stiffness, of the finite element representation. In contrast, in actual physical behavior, plate and shell structures behave in such a way as to minimize the participation of higher degree deformation forms, since such forms would excessively absorb strain energy and thus would cause excessively high stiffness in structural behavior. The actual physical action takes place through small in-surface adjustments of the membrane displacements, which naturally work to reduce the strain energy by eliminating unnecessarily complex strain states. The membrane displacements which accomplish this are necessarily of a rapidly varying, i.e., high polynomial degree, type.

## Stability Element Formulation

The approach of Reference 3 was to introduce axial displacement forms for the beam element through the 5th degree polynomial forms. This provided a quartic axial strain which was used to "erase" the quartic nonlinear axial strain resulting from the square of the bending slope. The method employed to determine the amplitudes of the added axial displacement forms was minimization of the potential energy on the elemental level. This corresponds to the physical behavior by which the structure seeks a minimum energy state. The authors also showed that the same result can be obtained by directly constraining the axial strain to eliminate its high degree polynomial components.

The procedures of Reference 3 appear to be extendable to apply to plate and shell elements. The basic approach is to start with an available element which has been used for linear analysis, to add supplementary membrane (HMN) functions to control the high degree nonlinear membrane strains, and to derive appropriate constraint equations to effect this control. This was done with the BCIZ and

AZI (References 5, 6) elements. Considerable additional complexity occurs, for several reasons: there are three membrane strains to control and two displacement components to deal with, rather than the single axial strain and displacement of the beam problem; constraints on the added displacement forms must avoid creating inter-element incompatibilities; the strain constraints must be formulated in two dimensions, and the equations are difficult to deduce. The overall procedures required to deal with the plate and shell elements is described in detail in Reference 1. Briefly, it is as described below. The development is based on extensions of the triangular BCIZ (Reference 5) and quadrilateral AZI (Reference 6) elements. Double curvature and fully coupled membrane and bending displacement states are included.

- Higher degree polynomial forms in the membrane strain equations, resulting from large deflection-induced nonlinearities, and also from initial curvature, for shell elements, are determined. Thus, amplitudes, in terms of nodal bending freedoms, of polynomial forms such as $x^2y$, $xy^2$, $x^3$, $y^3$, $x^2y^2$, etc., are determined.

- Supplementary membrane displacement forms are deduced such that the added forms are able to produce these same polynomial strain terms. The supplementary membrane functions must form a complete set, in combination with the basic membrane functions of the element. The high degree polynomial strain terms must be produced as independent functions.

- A set of strain constraints is developed to reduce the polynomial degree of the membrane strains to the same degree which they have in the basic element formulation. The constraints must avoid inter-element incompatibility while providing as complete as possible a control over the high degree membrane strains. Dependencies among the strain constraint conditions must be avoided; this can be a troublesome problem.

- Those supplementary (HMN) membrane functions which do not participate in the above constraints, but which are present in order to keep a complete set of functions, are reduced out by potential energy minimization at the elemental level. The minimum energy constraint cannot be used for HMN functions at inter-element boundaries, because inter-element displacement incompatibilities would result.

In applications, the above has led to procedures in which the added displacements include high degree forms defined independently along the sides of the elements and in their interiors, and in which the constraints include both explicit strain constraints and minimum potential energy constraints, all on the elemental level. When the added displacements are parallel to a particular line (say, a side) in the element, the amplitudes of the added displacements are completely fixed by the values of the bending displacement along that line, with numerical values defined by the constraint equations. This is the case for most of the HMN constraints used in the two elements, and assures inter-element compatibility of displacements parallel to the sides of the elements. For the triangular element, HMN functions normal to the sides of the elements were found to cause inter-element incompatibilities, and hence were dropped from consideration. In the numerical work to date with the quadrilateral element, supplementary membrane displacements normal to the sides of the elements have in some cases been used. These functions serve to eliminate high degree nonlinear membrane shear strains. The element derivation contains such normal-to-the-side displacements of cubic and quartic variations. It has been reasoned (Reference 1) that the cubic form would produce unacceptable inter-element incompatibility, and these functions are not retained. The quartic form was felt to be acceptable, and is currently retained in calculations. The inter-element incompatibility associated with the quartic function is nonzero but generally very small.

Figure 1 illustrates the manner in which the explicit strain constraints are imposed for the two elements in the computer programs. For the triangle, the extensional strain parallel to the three sides, designated by $\varepsilon_s$, is controlled as a polynomial function of the side-length coordinate, s. This control is through the 4th degree polynomial. For the quadrilateral, the extensional strain parallel to the edges and the mid-line of the element, in both the x and y directions, are controlled through the 2nd degree polynomial form. Thus, $\varepsilon_x$ is controlled along lines 1-3, 8-4, 7-5, and $\varepsilon_y$ along lines 1-7, 2-6, and 3-5 in the figure, using, respectively, cubic functions for u and v. In addition, the shear strain $\varepsilon_{xy}$ is controlled for the cubic polynomial form along the four sides of the element. This is accomplished through added displacements of quartic form, normal to the sides of the elements, as discussed above.

All HMN constraints are formulated for the parent element of the isoparametric family. This is essential because the displacement functions are expressed in terms of the parent element coordinates. It is essential to account for the general isoparametric element in the HMN constraints. This is not conveniently done through the normal Jacobian transformation of the isoparametric family, since distinct polynomial forms, rather than numerical values at integration points, are required for the HMN constraints. Consequently, it was necessary to re-derive the HMN constraints, dealing at the outset with the general isoparametric element. This was done through the use of element side and mid-line (see Figure 2) length scale factors and similar scale factors relating to differentiations.

## Coordinate Systems and Updating

The application of the HMN strain constraint procedure and the intended purposes of the new elements have implications regarding coordinate systems and stepwise updating. The elements are to handle large rotation problems and are to avoid cumulative error by computing total nonlinear strains rather than summing increments. These requirements are best satisfied by using element coordinate systems which follow the elements throughout the deformation. If a fixed system, such as a fixed cartesian or fixed shell surface intrinsic system were used, and the stepwise calculation implemented by updating the element position within this system, a problem of role exchange between the element displacement forms would occur. In this role exchange, the displacement forms (e.g., cubics) intended for the bending displacements, and, say, directed along a locally fixed z-axis, would after a large rotation be partly directed along material lines lying in the plane of the deformed element. Conversely, the membrane displacements along locally fixed x and y axes would, after a large rotation, be partly directed along a material line normal to the deformed element. This difficulty limits the allowable magnitudes of the rotations for elements with distinctly different membrane and bending displacement forms, in formulations using fixed coordinate systems for displacement definitions. It suggests that "following" coordinate systems with updating be used for nonlinear elements.

A second factor reinforces this conclusion in the case of the stability element approach. The nonlinear strain equation for $\varepsilon_x$, referred to locally cartesian coordinates, is

$$\varepsilon_x = u,_x + 1/2 \, [\underline{u,_x^2} + \underline{v,_x^2} + w,_x^2]$$

The term $w,_x^2$ is important even for small rotations; $u,_x^2$ and $v,_x^2$ become important terms for large rotations. As discussed earlier, the membrane functions u and v contain high degree polynomial forms, to implement the HMN methodology. If the underlined terms in the above equations are retained, these high degree components of u and v will create undesirably high degree poloynomials in $\varepsilon_x$. This is precisely the effect which the HMN procedure is intended to avoid. Consequently, in formulating the HMN elements, it appears to be necessary to limit the magnitudes of the rotations through the use of coordinate systems which follow the elements throughout the entire deformation process, and correspondingly to omit terms in the strain displacement equations of the type underlined above. A system of cartesian, updated, element baseplane coordinate systems was developed for this purpose. The rotations which occur during a given load step are small, and after the step has been computed, and the cartesian baseplane coordinate systems updated, the total rotations of the material elements relative to their updated coordinate systems remain small. Thus, total strains can be computed accurately with all nonlinear terms of the types underlined above omitted from the membrane strain equations. In use for finite element analysis with reasonably small elements, this approach guarantees that element material slopes will always be small relative to the baseplane of reference. Hence, it is possible to use a shallow shell type of deformation definition, with its attendant simplification of formulation. This is a significant gain for the nonlinear elements under consideration, since nonlinearities develop through accumulated element slopes rather than through cumulative changes of the shell geometrical parameters, referred to, say, a lines-of-curvature coordinate system.

The baseplane following coordinate system procedure is illustrated by Figure 2. In this figure, coordinate axes ($x_o$, $z_o$) are used to set up the equations for the first load step, which terminates with the element in the position described by the updated axis system ($x_1$, $z_1$). Similarly, load steps two and three cause displacements to the positions described by the updated systems ($x_2$, $z_2$) and ($x_3$, $z_3$), etc. In updating the coordinate systems in this way, the total displacements of the element are transformed to the new axes, with only the rigid motions removed. Used in conjunction with a Lagrangian strain definition, this approach is best termed a "start-over-total-Lagrangian" approach, since total strains are repeatedly computed from new starting orientations. The displacement transformation is not a straightforward geometrical transformation of displacement increments in the usual sense. Instead, it is a special transformation based on the idea of computing total element displacements, referenced to the updated coordinate system, measured between the deformed element position and an undeformed element suitably mapped onto the updated system. This procedure assures that total strains can be correctly computed using the transformed displacements referred to any updated coordinate system. Figure 3 illustrates the basis of the transformation for the simple case of a beam element. An element is shown referred to an *initial* coordinate system ($x_o$, $z_o$) and, after a deformation step, to an updated system ($x_1$, $z_1$). The undeformed shape of the element is shown on both coordinate systems; the element end points always lie on the x-axis. Displacements are defined to be the vector differences between points on the deformed and undeformed elements, referred to the particular coordinate system in question. This definition suffices for the calculation of strain. The element in system ($x_o$, $z_o$) has accumulated displacements which, through prior transformations, are referred to this system. Thus, for a point P, the accumulated displacements can be denoted by the vector notation

$$(\overrightarrow{u_{op}, w_{op}})$$

In which the subscripts denote both the reference system and the point P. The incremental displacements which occur during the present step are also referred to the ($x_o$, $z_o$) system, and are

$$(\overrightarrow{\Delta u_{op}, \Delta w_{op}})$$

The position vector of point P on the undeformed element in the $(x_o, z_o)$ system is denoted by

$$(\overrightarrow{x_{op}, z_{op}})$$

Thus the point P after the current step has the position

$$(\overrightarrow{x_{op}, z_{op}}) + (\overrightarrow{u_{op}, w_{op}}) + (\overrightarrow{\Delta u_{op}, \Delta w_{op}})$$

Referred to the $(x_1, z_1)$ system, the position of point P on the undeformed element is denoted by

$$(\overrightarrow{x_{1p}, z_{1p}})$$

It is noted that the individual components satisfy $x_{1p} = x_{op}$ and $z_{1p} = z_{op}$, since these values refer to the undeformed state. The total accumulated displacements at the end of the step, referred to the $(x_1, z_1)$ system are

$$(\overrightarrow{u_{1p}, w_{1p}})$$

The definitions of these components are as shown on Figure 3. The position of point P can now be given in both coordinate systems. Equating these necessarily identical vector quantities gives

$$(\overrightarrow{x_{op}, z_{op}}) + (\overrightarrow{u_{op}, w_{op}}) + (\overrightarrow{\Delta u_{op}, \Delta w_{op}}) = \vec{R} + (\overrightarrow{x_{1p}, z_{1p}}) + (\overrightarrow{u_{1p}, w_{1p}})$$

where $\vec{R}$ is the translation of the origin. This equation is solved for $(\overline{u_{1p}}, \overrightarrow{w_{1p}})$, in component form, by employing the appropriate unit vectors of the two coordinate systems and forming dot-products with both sides of the equation. Differentation of the vector components $u_{1p}$ and $w_{1p}$, with respect to the updated baseplane coordinates, treated as material coordinates, yields transformed displacement derivative freedoms, if required, and also all of the quantities required for calculation of the total nonlinear strains. A detailed set of equations of this transformation procedure is given in Reference 1. It is seen that this system of treating displacements handles the element displacement functions as displacements in the directions of the convected baseplane coordinates. Each incremental displacement consists of element displacement function increments referred to an updated element baseplane.

There is an equivalent alternative to the above approach for the direct calculation of purely nodal displacements. This is to maintain cumulative nodal displacement values referred to a fixed global coordinate system, and use these values to transform back to the successively updated element systems. Such a procedure is a useful part of the solution procedure in any case, because the global values are a convenient means of determining the orientations of the successive element baseplanes. However, this procedure does not provide nodal displacement derivative freedoms, which are needed for some types of elements. Both methods are used, as appropriate, in the computer coding of the two elements under study in this contract. The triangular element (based on the BCIZ element, Reference 5) uses nodal derivative freedoms. For this element, the above described transformation was found essential for transforming the derivative freedoms. The quadrilateral element (based on the AZI element, Reference 6) uses nodal displacements and nodal rotations as freedoms. For this element, since rotations rather than derivative freedoms are used, a different type of transformation was required. This particular transformation was developed during the checkout of the quadrilateral element, and is outlined below.

Figure 4 shows a portion of a deformed element of the AZI type, in which transverse shear deformations are permitted. For simplicity a beam element is considered here. The displacement state of the element is referred to its successive baseplanes. The start-of-step baseplane is denoted here by the $x_0$ axis, and the end-of-step baseplane by the $x_1$ axis. The displacement quantities for the start- and end-of-step states, for a point P on the midline of the element and for the fiber PS, defined to be the original normal to the midline in the undeformed state, are indicated in the figure. The subscripts refer to the coordinate system to which the displacements are referred. The transformation described earlier in this section permits the calculation of $w_{1p}$ from the values

$w_{op} + \Delta w_{op}$. Here we are concerned with the determination of $\theta_{1p}$ from the values of $\theta_{op}$ and $\Delta\theta_{op}$. A procedure which immediately suggests itself to make this determination is a simple subtraction of angles, including the angle change between the successive baseplanes. This is not the procedure used, however. Instead, the method developed is based on representing the material vector $\vec{PS}$ in the successive coordinate systems and deducing the needed angles from its components. This method is convenient in the three dimensional case, and also incorporates the effects of the rotations on the z-variation of the bending displacement, w. This was found essential for accurate calculation of nonlinear strains. The transform step is briefly described here, since it is not considered in Reference 1. The method described here is used in the computer code for the quadrilateral element. A more exact procedure, which would be required for larger single step displacements or for shell elements joining non-tangentially, is described in Section 2.4. We assume as before that all local and single step angles and strains are small compared to unity and that $\theta$ and $\sin\theta$ can be taken as equal. Then the vector $\vec{PS}$ in the $x_o$ system is given by

$$\vec{PS} = |PS| \cdot \lfloor\, \theta_{opy},\ -\theta_{opx},\ \sqrt{1-\theta_{opy}^2-\theta_{opx}^2}\ \rfloor \cdot \begin{Bmatrix} \vec{\nu}_{oz} \\ \vec{\nu}_{oy} \\ \vec{\nu}_{ox} \end{Bmatrix}$$

where $|PS|$ is the original length of $\vec{PS}$, subscripts x and y denote rotations about the $x_o$ and $y_o$ axes, and the vectors $\vec{\nu}$ are unit vectors in the $x_o$ system. The assumption that the incremental angles of rotation are sufficiently small to be added in any order is implicit in this equation. The third component of the row vector retains the squared terms in order to included the effect of foreshortening of the normal due to its rotations. For simplicity we abbreviate this equation as follows

$$\vec{PS} = |PS| \cdot \lfloor\, \theta_o \,\rfloor \cdot \{\vec{\nu}_o\}$$

After the deformations of the current step,

$$\vec{PS} + \vec{\Delta PS} = |PS| \cdot \lfloor\, \theta_o + \Delta\theta_o \,\rfloor \cdot \{\vec{\nu}_o\}$$

$|PS|$ is unchanged because the condition of inextensible normals is imposed. This condition, and its use in determining the nonlinear transverse shear strain, discussed later, are the reasons for retaining the two squared terms in the $z_o$ component, $\sqrt{1-\theta_{opy}^2-\theta_{opx}^2}$ .

The vector $\overrightarrow{PS} + \Delta\overrightarrow{PS}$ is easily transformed to the $x_1$ system by means of a direction cosine matrix of the cartesian coordinate transformation $x_o \rightarrow x_1$. Denoting this matrix by $[\lambda]$, there results

$$\overrightarrow{PS} + \Delta\overrightarrow{PS} = \lfloor\ \theta_o + \Delta\theta_o\ \rfloor \cdot [\lambda] \cdot \{\ \vec{\nu}_1 \} = \lfloor\ \theta_1\ \rfloor \cdot \{\ \vec{\nu}_1\ \}$$

The definition of $[\lambda\rfloor$ consistent with the above equation is

$$[\lambda] = \begin{bmatrix} \vec{\nu}_{01} \cdot \vec{\nu}_{11} & \vec{\nu}_{01} \cdot \vec{\nu}_{12} & \vec{\nu}_{01} \cdot \vec{\nu}_{13} \\ \vec{\nu}_{02} \cdot \vec{\nu}_{11} & \vec{\nu}_{02} \cdot \vec{\nu}_{12} & \vec{\nu}_{02} \cdot \vec{\nu}_{13} \\ \vec{\nu}_{03} \cdot \vec{\nu}_{11} & \vec{\nu}_{03} \cdot \vec{\nu}_{12} & \vec{\nu}_{03} \cdot \vec{\nu}_{13} \end{bmatrix}$$

in which the indicated operations are scalar products and the subscripts of the unit vectors refer, respectively, to the coordinate system to which they belong, and their particular component direction, in the order x, y, z. The updated vector $\{\theta\}$, is thus given by

$$\theta_1 = [\lambda]^T \{\ \theta_o + \Delta\theta_o\ \}$$

and the needed values $\theta_{1py}$ and $\theta_{1px}$ are obtained as the first component and the negative of the second component of $\{\theta_1\}$. The matrix $[\lambda]$ is readily obtained in the computer code from the product of the transformations between the element baseplane coordinate systems and the global coordinate system, for the start-of-step and end-of-step conditions.

The bending displacements, w, are also transformed between the start-of-step and end-of-step systems. The result of this transformation, in conjunction with the known displacement shape functions of the element, permits calculation of the bending slope quantities $w,_x$ and $w,_y$, referred to the end-of-step system. In doing this, the original element dimensions and shape are used, consistent with the approach of using material coordinates and obtaining a nonlinear strain of the Lagrangian type. The element w-displacement functions are considered to be convected to the updated baseplane coordinate system, in order to determine the derivatives $w,_x$ and $w,_y$. The end-of-step values of $w,_x$, $w,_y$, and the rotations, $\theta_{px}$ and $\theta_{py}$, suffice to obtain the total nonlinear transverse shear strain. Further details of this calculation are discussed below.

## Shell Strain-Displacement Equations

A final item will complete this description of the conceptual basis of the two shell elements under study. It was noted earlier in this section that, to obtain a workable theory for stepping out nonlinear solutions, it is preferable to update the element baseplane in order to maintain a small angle relationship between the element and its reference system. The resulting availability of the updated baseplane strongly suggests the use of a shallow shell theory in which the displacements and forces are referred to the cartesian baseplane system rather than to the shell intrinsic curvilinear system. The simplifications obtained in this way are particularly helpful for nonlinear problems. Consequently, it was decided to base the element development on a shallow shell type of theory, using cartesian displacement definitions, with baseplane updating to permit extension to the large deflection regime. Since shallow shell theories do not as a rule use purely cartesian displacement definitions, it was necessary to derive a new set of equations, starting from basic principles. To do this, a tensor approach was used, and the strains were derived from the changes of the metric tensor between the undeformed and the deformed states. This is a superior approach, which naturally provides all of the nonlinear effects of large displacements, without recourse to the more conventional geometric-deductive procedures. The derivations for the triangular (Kirchhoff-type) and quadrilational (transverse shear strain included) elements are given in Reference 1. Numerical work has pointed to a very important result of this overall procedure. The use of cartesian displacements, specifically the w-displacement, leads to a much different definition of the direct membrane strains than the conventional theory (both deep and shallow shells) in which w is normal to the shell midsurface. Conventionally, the membrane strain includes terms such as w/R, where R is a shell midsurface radius and w is normal to the midsurface. This term is replaced, when cartesian displacements are used, by terms like $w,_x^o \cdot w,_x$, where $w^o$ is the initial w-shape of the shell element, referred to its baseplane, and the commas denote differentiation. For a given w distribution over the element, these two terms provide very different forms of contributions to the membrane strain. They are made equivalent by differing forms of the membrane displacement, u, which contributes the term $u,_x$ to the strain. It can be reasoned that, for conventional strain equations, used in a finite element application, the membrane displacements are called upon in part to supplement the bending displacement in order to achieve satisfactorily strain-free rigid body motions of the elements. This generally requires very competent membrane displacements, particularly for low energy deformations such as inextensional bending. For the cartesian-based strain equations, rigid motions are automatically obtained, and the membrane displacements are called upon to supplement the bending displacement in order to achieve smoothly varying membrane strain states. In both cases, the use of membrane displacements of higher polynomial degree than the bending displacements is required for accurate solution of a wide class of shell problems. All of the above pertains to the case of linear analysis, and is governed primarily by the effects of shell curvature. For nonlinear analysis, a similar situation exists except that the membrane functions must be sufficiently competent to handle large rigid motions and changes in shell geometry (curvatures, twist) in their efforts to achieve stress-free rigid motions and smoothly varying strain states. It is noted at this point that the HMN functions, which increase element membrane displacement competence, are beneficial for linear and nonlinear analysis with shell theories of either the conventional or the cartesian-based types. Example problem #1, Section 2.4, is a case in which the HMN functions are crucial to an accurate problem solution in the linear case.

In approaching the derivation of the shallow shell type of nonlinear strain equations, consideration was given to the results of Reference 7, in which it was shown that shallow and deep shell theories can give very different results for certain types of problems. It was decided that it is the absence of certain types of forces in the membrane, i.e., baseplane, equilibrium equations of conventional shallow shell theory which leads to significant errors in particular problem solutions (Reference 7). The terms in question are the products of the transverse shear forces with the shell midsurface slope angles. These products provide membrane-direction forces, resulting from the transverse shear forces. These are omitted in conventional shallow shell theory as a part of the basic assumption that the transverse shear forces are small. They can be either retained or omitted in the cartesian-based theory, depending on retention of certain small terms in the strain-displacement equations. The types of problems in which the error due to this omission can be large are those in which there is strong bending combined with small membrane stress levels, either over the whole shell or in critical local regions. It would appear that nonlinear large deflection behavior would accentuate these errors, for two reasons: large deflections are most likely to occur in strong bending rather than strong membrane deformation problems; the rotations of the large deflection state will increase the importance of those force components which are conventionally ignored in the shallow shell theory. Consequently, shell equations were derived which retain the simplifications of shallow shell theory without omitting these particular terms in the equilibrium equations. The terms which were retained in the strain-displacement equations are in the definitions of bending and twisting deformations, and involve products of membrane displacement gradients and shell curvatures and twist. Both initial and subsequent curvature and twist are involved, so that the added terms serve to incorporate nonlinearities into the bending and twisting moments. Details of this derivation and the strain-displacement equations are given in Reference 1.

During the numerical evaluations of the quadrilateral element, it was found that physically unexplainable large residual lateral forces and transverse shear strains occurred in nonlinear bending problems. The cause of this behavior was traced to certain omitted nonlinear terms in the transverse shear strains, as derived in Reference 1. The original derivation included the products of the rotations with the membrane displacement gradients, but omitted terms involving products of the bending slopes with the z-direction derivatives of the bending displacements, i.e., the terms $w,_z \cdot w,_x$ and $w,_z \cdot w,_y$. These terms were omitted on the basis that all displacement z-derivatives were dropped in the original derivation, through direct use of the rotations of the normals, $\theta_x$ and $\theta_y$. However, it can be easily shown that the retained nonlinear terms, $u,_x \cdot \theta_x$, $u,_y \cdot \theta_x$, $v,_x \cdot \theta_y$, and $v,_y \cdot \theta_y$, are in most problems very nearly cancelled by the omitted terms, $w,_z \cdot w,_x$ and $w,_z \cdot w,_y$, in the nonlinear strain equations for the transverse shear strains. Thus, the omission of the latter terms resulted in large nonlinear contributions to the transverse shear strains, producing corresponding large residual shear forces. Coupling with the bending slopes caused serious loss of accuracy and convergence difficulties in nonlinear problems. When the omitted terms were included, the transverse shear strain was reduced to reasonable values, and the difficulties with the residual loads were eliminated.

To retain the effect of $w,_z$ in an element which retains only the freedoms u, v, w, $\theta_x$, and $\theta_y$ is a somewhat difficult task. The value of $w,_z$ was constructed from the condition of inextensional normals and the values of $\theta_x$ and $\theta_y$, and has the form, for small incremental rotations,

$$w,_z = -\frac{1}{2}(\theta_x^2 + \theta_y^2)$$

Hence, the terms $w,_x \cdot w,_z$ and $w,_y \cdot w_{\,z}$ are cubic in the displacement magnitudes. The basic strain formulation of the elements (Reference 1) is of the second degree in the displacements, and does not readily permit including the cubic terms. Consequently, the added nonlinear terms were included in the calculation of strains, stresses, and residual loads, but not in the formation of the element stiffness matrices. This type of approximation may slow the convergence of stepping/iterative calculations, but does not affect the accuracy of converged solutions. In future work it would be desirable to include the extra terms at the stiffness matrix generation stage of the calculations.

## 2.2 Computational Procedures

This section outlines the sequence of the computational procedures used for the two shell elements, with particular emphasis on those calculations which are unique to the stability elements and to the coordinate system updating used. Details such as equations and matrix definitions are given in Reference 1.

The purpose here is to describe the overall scheme and magnitude of the computational effort.

Coordinate Systems

Each element uses three principal coordinate systems: the baseplane system; the "solution" systems, which are cartesian nodal systems; and the global system. These are illustrated by Figure 5. The solution systems are nodal triads which are averages of the joining element baseplane systems. During a single solution (load) step, a single solution system is used without updating, for each node point of the structure. The solution systems are updated at the outset of each solution step. Iterative corrections to the equilibrium state are made within each solution step, based on residual loads evaluated at each iteration. Each residual load evaluation utilizes an updated element baseplane for the evaluation of the strains, stresses, and residuals. That is, the displacements of the previous iteration are used to update the element baseplane, the total deformation state is transformed to the updated baseplane, and total nonlinear strains and stresses are thereby computed. The virtual work integration, for the residual load evaluation, uses virtual displacement increments which are likewise referred to the updated baseplane, and are increments from the total deformation state referred to that baseplane. The repeated updating of the baseplane systems at each iteration is done to assure an accurate calculation of the total Lagrangian strain, even though the displacements and rotations of the step may be large. This is probably not necessary for well-posed stepwise loadings, and may be a candidate for code simplification in the future. The final updated baseplane systems of a given load step, i.e., those corresponding to the converged solution for the step, are the start-of-step baseplane systems for the next load step. Figure 6 illustrates schematically the use of these coordinate systems in a two step problem. The figure also gives the names of the transformation matrices between the coordinate systems, for later reference, and indicates thereby which transformations require updating for new load steps and for iterations within a single load step.

For the triangular element only, one other transformation of a coordinate system type is required. This transformation makes the transition between the so-called "deformational" freedoms of the original derivation of this element (Reference 5) and the baseplane-referenced freedoms which are used in the transformations [TD] (see Figure 6). It is called "TSTAR."

For the quadrilateral element only, the conventional isoparametric transformation is used to determine the strains of the general element from the shape functions and derivative formulas of the parent element referenced to its cartesian coordinate system. This transformation does not appear in Figure 6 because it occurs at an earlier stage of the calculations.

Stiffness Matrix Transformations

The stiffness matrices of the elements are initially derived in the baseplane coordinate system and include all freedoms of the element. For the quadrilateral, these total 58 freedoms, of which 40 are nodal freedoms which are retained for the final problem solution, and 18 are HMN freedoms, of which 10 are eliminated by explicit strain constraints, 4 are deleted to avoid probable inter-element incompatibilities, and 4 are eliminated by a minimum energy constraint. All constraints are on the elemental level. For the triangular element, there are a total of 51 freedoms, of which 27 are nodal freedoms which are retained for the final problem solution, and 24 are HMN freedoms, of which 6 are eliminated by explicit strain constraints, 6 are deleted to avoid verified inter-element incompatibilities, and 12 are eliminated by a minimum energy constraint. All constraints are at the elemental level. The stiffness matrix transformation sequences are shown on Figure 7 for the two elements. The designations on the figure have the following meanings:

HMN — This is the transformation which imposes explicit constraints on the higher polynomial strain terms. It operates on the stiffness matrix as a conventional generalized coordinate transformation of the form $C^TKC$.

DELETE — This is a simple removal of rows and columns to eliminate freedoms which might cause inter-element incompatibilities.

MPE - This is a conventional stationary potential energy reduction performed on the elemental level.

TSTAR - This is a generalized coordinate transformation performed to change from the BCIZ "deformational" freedoms to freedoms which have a consistent sign convention and permit rigid body motions of elements.

TD - See Figure 6. This puts freedoms into the solution coordinate systems.

MERGE - This is a conventional merge of elemental freedoms to obtain structural equilibrium equations.

The stiffness matrix is formed at the outset of each load step, as indicated on Figure 6. The solution proceeds with iterative corrections within the step until either convergence is obtained or an input iteration limit is reached. In the latter case, the stiffness matrix is reformed, and all transformations performed again as indicated on Figure 7. The stiffness matrix formation and transformation accounts for about 75% of the computational time on the small problems studied to date.

Loads Transformations

The elements in their present form accept only nodal load inputs. These are input in the global system, and can be used directly in problem solutions after a coordinate transformation using TCAP (see Figure 6). However, the computation of the residual loads for iteration is done with all of the element freedoms, and the resulting load vector must be transformed and merged through the same steps used for the stiffness matrix. Thus, Figure 7 applies also to the transformation of the elemental residual loads. It should be noted that the elemental residual loads are computed with reference to the most current updated element baseplane. Hence, the TD matrix used (Figure 7) must be the most recent update of that matrix. The MPE transformation of the loads is the conventional one which creates a partial load vector which is saved, to be used later in the back-substitution solution of the displacements.

The loads transformations are done for every iteration within the load step, in contrast with the stiffness matrix calculations, which are done much less frequently.

## Displacement Transformations

The incremental solution of a load step or an iteration is initially referred to the solution coordinate systems (Figure 5, 6). A number of transformations and other calculations are made on these incremental values, as shown by Figure 8. The figure shows the incremental solution, consisting of nodal displacement quantities and called $\Delta q_{\text{solution coords}}$, as the starting point of the data reduction procedure. The calculations initially proceed along three separate paths: (1) TCAP is used to transform the increment to global components and to update the global coordinates, element baseplanes, and transformation matrices associated with the baseplanes; (2) the incremental solution is transformed to the start-of-step baseplane, using the old TD matrix, and then summed with the prior accumulated nodal displacements referred to this baseplane; the total displacements are transformed by means of the RDOT matrix to obtain total nodal displacements referred to the updated baseplane; (3) the incremental nodal displacements referred to the start-of-step baseplane are used in back-substitution to obtain the minimum potential energy (MPE) incremental freedoms, $\Delta\alpha$, which are then summed to form the running total of these quantities. At this point the calculation paths merge and the explicit strain constraints (HMN conditions) are imposed, using total rather than incremental strains, and directly computing total HMN freedom values. The results at this point include the accumulated nodal displacements $q$, referred to the updated baseplanes, and the total accumulated MPE and total HMN freedoms, $\alpha$ and $\beta$. These data suffice to compute strain, stress, and residual loads. The loads are transformed as discussed above, merged to form overall structural residuals, and tested for convergence. If convergence has not been obtained, a new increment of nodal displacements, designated as $\Delta(\Delta q)_{\text{solution coords}}$, is computed, using the same solution coordinate system and stiffness matrix as were determined at the start of the step. If convergence has been obtained, the solution coordinate systems are updated, and new TCAP and new stiffness matrices are generated. In addition, a number of data arrays are saved, as indicated on the figure. In some cases, if convergence is slow, the stiffness matrices are updated without obtaining convergence. In this case, since the residual loads remain referenced to the old solution coordinate system, the latter, and TCAP, are not updated. This particular option amounts to a forced "yes" answer to the convergence test, except that the TCAP is not updated. The advantage of updating the stiffness matrix is due to properly accounting for the effect of element deformed shape on element stiffness, and to accounting for element orientation (updated baseplane) relative to the solution coordinate systems.

The HMN transformation, as used in stiffness matrix transformation, is derived for use by incrementation. However, the actual calculation of the HMN freedoms is a total rather than an incremental calculation. Prior to the data processing of Figure 8, HMN constraint matrices in terms of total rather than incremental strains are formed, by simple changes in the earlier-generated incremental matrices. Total HMN freedoms are then calculated, avoiding cumulative error in these freedoms. For the quadrilateral only, the HMN matrices are also updated prior to this calculation, to account for the total bending deformation accumulated to the current point of the iterative calculations. The need for this updating is due to the fact that the quadrilateral basic displacement shapes are only second degree forms. Hence, the basic membrane strain states are linear, and cannot compensate for nonlinear effects due to the simplest, i.e., constant curvature and constant twist, lateral displacements. The triangular element, on the other hand, having basic displacement forms which are cubic, can at least partially compensate for nonlinear effects due to the simplest bending deformations with its basic membrane strain states. The updating of the HMN matrices was not necessary for the triangular element.

## 2.3 Solution Procedures

Solution procedure development has required a large amount of effort in the present research, even though the primary aim of the work has been element evaluation and improvement. The cause of this is in the nature of the residual loads. In elements of the types under study, in which nonlinear strains due to relatively large bending displacements are included during each solution step, very large membrane stresses and membrane residual loads occur. In combination with the bending slopes, these residuals create residual bending loads and moments which usually completely dominate the bending behavior of the structure. This behavior becomes more severe as the plate or shell becomes thinner, corresponding to the increasing ratio of the membrane stiffness to the bending stiffness. In general, the bending residual loads are in the proportion $(\Delta/r)^2$ to the applied bending loads, where $\Delta$ is the deflection magnitude and r is the section (or plate/shell) radius of gyration. The bending residual loads are distributed over the structure, among the different elements, according to the local magnitudes of the slopes. Hence, in performing a residual load iteration, the bending displacement adjustment tends to be much larger than the initial displacement, and is distributed quite differently over the structure. If the iterative adjustment is accepted at its computed magnitude and distribution, generally a grossly distorted deformation state and greatly increased residual load values result. This type of behavior almost always causes divergence of the solution procedure. This situation is in marked contrast to the behavior which occurs in analysis approaches in which residual loads are either not computed, or are computed by approximate methods which effectively omit the effects of the nonlinear straining which occurs during the increment. In these approaches, the residual loads are small, and convergence is generally rapidly achieved. However, for strongly nonlinear problems, such approaches generally yield solutions which diverge increasingly from the correct solution, as the magnitudes of the nonlinearly-induced strains increase.

The simplest way of improving solution procedure behavior, in fully nonlinear analysis, is to scale the magnitude of the iterative correction increment. Procedures often are coded with factors such as 0.5, for example, to be applied to these increments. Though this may at times be successful, it is an unsatisfactory method in two ways: the proper scale factor generally varies widely during a complete problem solution; the method fails to address the fact that the iterative increment is distributed incorrectly over the structure. These two difficulties are the principal obstacles to achieving a rapidly converging stepwise/iterative procedure for nonlinear analysis. It is required to achieve "intelligent," programmed procedures for controlling the size of the iterative increment, and also for controlling its "shape." The concept of "shape," as used here, includes both the distribution of deformations over the entire structure, and also differences in deformation magnitudes between different types of deformation, e.g., membrane as opposed to bending motions, etc. The present research has developed primitive, but effective, procedures for dealing with these two difficulties, and has thereby achieved convergence in problems which, in earlier, conventional versions of the solution procedure, were not solvable.

Subsequent paragraphs will describe the developed solution procedure. However, it is worthwhile to digress at this point in order to consider the implications of this work with regard to stepwise-linear solution procedures in general. It has been demonstrated by the solution procedure work of this research that both magnitude and direction are incorrect in stepwise-linear solutions of nonlinear problems. The errors begin with the initial application of load, and continue through all iterations and subsequent load steps. It has also been verified that updating the stiffness matrix tends to alleviate these difficulties. Clearly, the basic source of the error is in the stiffness matrix itself, both as regards overall stiffness magnitude, and as regards stiffness coupling effects between different deformation types, particularly between membrane and bending displacements. This total problem would be solved by using a nonlinear stiffness description of the structure, i.e., by using a stepwise-nonlinear approach for the solution of strongly nonlinear problems. Such an approach is available, and is described in References 8 and 9. It is termed the "static perturbation" method, and achieves fully stepwise-nonlinear performance without a great deal more computational effort than the conventional stepwise-linear approaches. For many problems, the stepwise-nonlinear approach would very likely be less expensive as well as more reliable, since fewer iterations and fewer stiffness matrix updates would generally be required.

Solution Procedure--Convergence Acceleration

The overall solution process consists of a set of load steps, within each of which is a set of iterations, each of which is based on the residual loads corresponding to the immediately previous displacement state. This latter state may result from either an input load step application or an iteration for a residual load application. The first significant improvement in convergence resulted from performing iteration steps which alternately permitted all structural freedoms to respond, or, alternatively, permitted only the membrane freedoms to respond. In this case, the membrane freedoms are those approximately parallel to the plate or shell surface, as defined by the solution coordinate systems (Figure 5). The effect of the membrane freedom iteration was to allow the structure to relieve much of the nonlinear strain and stress induced by the previous bending displacement increment. This nonlinear strain and stress relief serves to greatly reduce the magnitudes of the residual loads, with the result that the following (all freedom) increment will have much smaller bending displacements, and, hence, much less further nonlinear strain and stress generation. The purely membrane increment can be thought of in two ways: as a post-increment correction of the "shape" error of the previous all-freedom increment; and as a plausible physical action, which a plate or shell structure would naturally undertake to relieve equilibrium imbalances and achieve a reduced potential energy level. These are distinct, but, of course, basically equivalent interpretations. This solution procedure feature influenced deformation shape in a primary way, and, through residual load reduction, influenced increment magnitude secondarily. Convergence was thereby obtained for problems which had previously diverged except for very small load increments.

It was found, however, that for some problems the residual loads still tended to be large enough that convergence was not obtained except for small load steps. These problems were generally those with more elements or problems in which the overall structural stiffness depends strongly on displacement magnitude (e.g., the un-prestressed membrane problem). The basic difficulty was felt to be in the magnitude of the displacement increments, and how they were distributed over the structure.

To improve this aspect of the convergence, a procedure was implemented in which, in the all-freedom iterations, the amplitude of the increment was arbitrarily varied over a certain range, say 50% to 150% of the computed value, and a state of minimum residual was sought. For this purpose a measure of the residual based on a root-sum-square over all the residuals was used. The minimum was sought on the basis of a quadratic fit of the residual-measure versus the amplitude factor. This procedure performed well several times, but in general had the characteristic of extremely slow convergence. The reason was found to be as follows: the residual-measure for all amplitudes of the all-freedom increment (say, 50%, 100%, and 150%) generally exceeded the value of the residual-measure of the previous membrane-only increment, in rough proportion to the magnitude factor. Thus, a minimum residual was not found. The solution procedure was coded to reset the range of the search, to, say, 25%, 50%, 75%, and repeat the calculation. Generally, the failure to find a minimum repeated itself. Ultimately, the procedure accepted a very low amplitude factor for the all-freedom increment, and thus failed to make appreciable progress toward convergence.

The basic cause of these difficulties was the "shape" of the all-freedom increment, specifically the incorrect amplitudes of its membrane as compared to its bending freedoms. To remedy this, a repeated use of the membrane-only increment was implemented as follows:

> For each amplitude of the all-freedom increment, (say, 50%, 100%, 150%), residual loads are computed, and, based on these new residuals, a membrane-only increment is computed and added to the factored all-freedom increment.
>
> Total displacements, residual loads, and the residual-measure are recomputed for this "double increment", and this residual-measure is identified as belonging to the particular amplitude factor used.
>
> The search for a minimum residual-measure is done using these "hybrid" residual load states and residual-measures.
>
> The chosen amplitude factor is used for a final, fourth-time calculation, of the displacement increment and the residual loads. This calculation also uses a membrane-only substep.

In using this procedure, the single, membrane-only increment is no longer necessary and it was removed from the solution procedure. The double-search calculations are done for each load increment application as well as for the residual load iterations.

This procedure has been successful on all problems to which it has been applied. It is by no means a fully optimized procedure, and it is clearly primitive and costly. Nevertheless, it has provided convergence for previously badly divergent cases. It is felt that the principal importance of the procedure is that it verifies that increment "shape" control and amplitude control are both required, on a per-increment basis, to obtain convergence to a general class of strongly nonlinear problems. This has led to the strong conviction that some type of stepwise nonlinearity, based on a stiffness matrix which varies within each increment, is essential to cost-effective solution procedures for strongly nonlinear finite element analysis.

Several other items were found to be necessary for convergence, in addition to the overall method described above:

The stiffness matrix requires frequent updating, particularly while relatively large bending displacement increments are occurring. This greatly improves the quality of the computed increments.

The stiffness matrix updating includes a re-calculation of the geometric stiffness matrix. In this calculation, the stresses of the previous converged steps are used until the membrane stress state has recovered from its initial large, nonlinearity-induced, excursion. The residual-measure magnitude is used to control this decision process.

The residual measure has been modified to consider only bending direction ($w$, $\theta_x$, $\theta_y$) residuals. This was found essential to prevent the search (optimization) procedure from optimizing toward zero membrane residuals at the expense of the bending residuals. This did occur, and caused convergence toward very small displacement amplitudes, in some cases.

The iterative increment is checked numerically to assure that it does not exceed the basic increment magnitude of the load step. If it does, it is scaled overall, prior to the search calculations.

If the search calculations do not find a minimum residual-measure, the appropriate end of the search range is accepted as a satisfactory amplitude factor. This has proven satisfactory, and saves computation time compared to a complete search range change and re-calculation.

Increment rotation magnitudes are checked, and the increment is prevented from rotations large enough to violate the incremental small angle assumption (about 20°).

Solution Procedure--User Input

Because different types of nonlinear problems are best handled with somewhat different calculation sequences and details, certain data can be input by the user, as controls over the solution process.

The degree of refinement required for converged solution states is an input item. Too small a tolerance on this value can increase computer costs excessively.

The number of iterations to be performed before the stiffness matrix is updated is controlled by user input. Thus, for example, a set of values such as 1,2,2,3 specifies that, after the load step application, the stiffness matrix is updated immediately. Thereafter, two iterations are performed, the stiffness is updated, two more iterations are performed, etc.

The total number of permitted stiffness matrix updates is specified by the user. Since the updates are major computer cost items, this provides user control over costs of solutions which, for some reason, are converging too slowly.

The search range for the "double-search" procedure is controlled by user input values of the center point and the range on either side over which the search for minimum residuals will be performed. Thus, for example, the values of .80 and .35, respectively, will cause the search for the minimum residual state to take place with amplitude factors .45, .80, and 1.15. This control is particularly useful for structures which are known to stiffen or soften markedly as they accumulate deformation.

In order to handle problems with either "following" types of boundary conditions, or with conventional boundary conditions, the user can input particular fixed axes with respect to which boundary conditions will be imposed. These are referred to the global system. Alternatively, the solution coordinate system triads, which are convected with the deformation can be used to enforce boundary conditions.

In addition to the above, of course, the user may elect to insert zero load steps. These steps cause a general cleanup of all coordinate system transformations and data updating. This has not been found necessary in the problems solved to date, with the most recent solution procedure.

Finally, the program has been structured to allow operation in the HMN-mode, or as a conventional nonlinear program (using the basic AZI and BCIZ elements). This has permitted comparisons of HMN and non-HMN element performance.

## 2.4 A Consistent Transformation for Finite Rotational Freedoms

With isoparametric shell elements of the AZI type (Reference 6), or with corresponding types of curved beam elements, a general finite element model will have three translational and three rotational freedoms at each mode. The three, rather than two, rotational freedoms at each node are required in order to handle elements which intersect at other than 180-degree angles. (Three rotational freedoms are also required if beam torsional behavior is to be included.)

The translational freedoms may be transformed between the solution and element-baseplane coordinate systems, using a simple 3x3 matrix of direction cosines, and that transformation may be applied exactly to finite incremental or cumulative translations. In the case of finite rotations, however, the resulting configuration is dependent upon the order in which the rotations are performed.

This order dependence cannot be considered when incremental rotations are calculated during the usual matrix solution procedure (there the rotations are considered to be infinitesimal). After a series of such rotation increments are calculated, transformed from solution to baseplane systems, and summed to cumulative values, an inconsistency develops relative to the deformed configuration of adjacent elements. That is, a cumulative error is created which is equivalent to the existence of physical gaps (slope discontinuities) between elements. This inconsistency affects the residual-force equilibrium computations, so that it cannot be eliminated by the iteration procedure.

In order to develop a consistent transformation for the rotational freedoms, imagine that at each node the following entities exist.

1) a small, rigid globule of material

2) an arbitrarily defined solution-coordinate-system triad $\hat{X}$-$\hat{Y}$-$\hat{Z}$

3) a unit normal vector $\hat{N}_I$ for each (Ith) element joined to this node (the initial orientation of $\hat{N}_I$ in the undeformed configuration is normal to the Ith element baseplane)

These three entities are rigidly attached together, so that they translate and rotate as a single rigid body during successive load increments. Figure 9a illustrates the concept for the simple case of a node where three curved-beam type elements are joined. The three entities and the adjoining elements are depicted there in the initial undeformed configuration. Figure 9b shows one of the elements in a later deformed state, with a new vector N which we define as being a unit normal to the deformed baseplane. Because $\hat{N}$ has rotated with the rigid material globule at the node, and this rotation is in general different from that of the element baseplane, the vectors $\hat{N}$ and N are not parallel. The difference between these two vectors ($\hat{N}$-N) will provide a convenient measure of the element nodal rotations.

Specifically, the consistent element rotations are determined according to the following steps.

1) Given start-of-increment solution coordinate triad $\hat{X}^o$-$\hat{Y}^o$-$\hat{Z}^o$ at a node. This triad can be used as the base vectors for expressing other vector quantities.

2) Perform the matrix solution procedure to obtain nodal incremental translations and rotations (expressed in terms of $\hat{X}^o$-$\hat{Y}^o$-$\hat{Z}^o$ system triad).

3) Using the nodal translations, determine new element baseplane orientations and associated normal vectors N.

4) Using the nodal incremental rotations, calculate new end-of-increment solution coordinate triad $\hat{X}'$-$\hat{Y}'$-$\hat{Z}'$ at a node. (This calculation is somewhat arbitrary due to the order dependence of finite incremental rotations. However, the new triad will be used consistently for all elements joining the node, so that any error introduced will be eliminated by the residual force iteration.

5) Since $\hat{N}$ is rigidly attached to the solution triad, it is given in terms of $\hat{X}'$-$\hat{Y}'$-$\hat{Z}'$, and can then be transformed to the base vectors $\hat{X}^o$-$\hat{Y}^o$-$\hat{Z}^o$.

6) Cumulative element nodal rotations are then computed from the difference ($\hat{N}$-N), by taking its vector dot product with the element baseplane coordinate axes.

This procedure allows the calculation of cumulative element nodal rotations, which have no cumulative error. That is, they are consistent among all elements adjoining the node, so that no slope discontinuity effects are allowed to accumulate.

## 2.5 Discussion of Numerical Results

The triangular and quadrilateral HMN elements have been used to solve a large number of simple problems. The early numerical work dealt principally with the triangular element, and demonstrated that the performance of this element was satisfactory, both in overall element linear and nonlinear behavior, and also in regard to the HMN-related behavior. Example #5 of this section illustrates this work. It was decided on the basis of this work and early experience with the quadrilateral element that the latter is the better element of the two by a wide margin. Consequently, the majority of HMN-element evaluation was done with the quadrilateral. This work is covered by examples #1 through #4 of this section. The basic difficulty with the HMN-BCIZ element is believed to be related to a known deficiency of the basic BCIZ element, namely, that it fails to maintain inter-element slope conformity.

The goals of the numerical evaluations are as follows:

> To understand the basic behavior of the elements, considering particularly the influences of coordinate systems, the HMN function activity, and the effects of initial curvature.
>
> To compare the behavior of the HMN-elements with non-HMN-elements, considering both large and small deflections, and both initially flat and initially curved elements.
>
> To develop solution procedure concepts and methods suitable to provide good convergence properties for problems using HMN-elements.

These goals have been met satisfactorily, and a thorough evaluation of the elements has been made. It has not been possible, however, to demonstrate the elements in nonlinear problems of practical importance (e.g., shell buckling), due to the solution procedure difficulties which were encountered and also the fact that the computer programs were designed for use with very few elements. It has also not been undertaken to verify nonlinear predictions against available exact solutions, despite the original intention to make such checks. Little additional effort would be required at this point, due to the convergent solution procedures developed, to make such checks for simple problems.

One of the conclusions reached in the evaluations of the HMN-AZI element is the desirability of a formulation change to a modified type of isoparametric element having displacements entirely nodally defined, with explicit HMN constraints on higher degree strain polynomials replaced by potential energy constraints. The bases of this conclusion are contained in the discussions of the numerical examples of this section, particularly examples #2 and #3.

Example 1: Pinched Cylinder

Figure 10 shows a cylindrical shell pinched by line loads 180° apart. To study this problem, a quarter-circle model using four AZI-HMN elements was analyzed, using the boundary conditions indicated. Both zero and non-zero values of Poisson's ratio were used. This problem was a valuable example in many respects. At the outset, it proved a severe test of solution procedures, and led to several improvements in these methods. In addition, the pinched cylinder problem has several particularly interesting features for the present investigation: (1) it is a sensitive indicator of the differences between shallow shell theory and deep shell theory; (2) it shows very strong effects of the action of the HMN freedoms; (3) it illustrates clearly the relationship between the use of the baseplane coordinate system and the HMN functions and constraints; (4) it illustrates a difficulty inherent in nonlinear plate bending analysis with nonzero Poisson's ratio; (5) it affords an opportunity to consider a case of nonlinearity of purely geometric (negligible nonlinear strain or stress effects) origin.

The radius of the cylinder is 1" and its width is 0.4". Two thicknesses were used: .025" and .100". Loads from .34# to 67.9# were applied vertically at the upper edge of the quarter-circle structural model, and both ends of the quarter-circle were restrained against rotation. Deflections at the load up to .098" were computed. This deflection value is 9.8% of the radius and 3.92

times the thickness of .025". At this deflection a noticeable softening of the structure, due to geometry change, has occurred, as compared to deflections for lower load levels. The apparent softening is estimated to be 15%, based on the deflection under the load. The non-HMN solutions yield deflections ranging from about 1/3 to 1/4 of the HMN values, depending on the displacement amplitude. A check of the computed displacements at very small load levels against the comparable analytical solution for ring bending shows that the four element array used in this problem is about 5.7% too stiff. This is a reasonable error for a four element array, using constant curvature elements, for a problem of this type. Figure 10 shows the deflected shape of the structure, plotted to scale, and the force-deflection curve, for the case of t = .025".

The stress resultants in this problem are in general quite rapidly varying over the 90° arc, and are strongly dependent on the number of elements used, the value of Poisson's ratio, the shell thickness (R/t), and the type of shell theory (deep or shallow) used. This type of complexity was found in the present study, and also in Reference 7 for the linear case. In general, finite element internal stresses cannot be compared directly with applied loadings, except for very small elements. Even in this case, often the stress resultants are difficult to interpret, except on the basis of some type of average value over entire elements. The reason for this lies in the fact that the finite element method achieves equilibrium with respect to generalized loads, which are work-equivalents of the stresses, rather than with respect to actual boundary values of the stresses themselves. In the present problem, evaluation of the stress resultants was therefore made qualitatively. Stress resultant values were compared for the HMN and non-HMN cases and for deep versus shallow shell strain equations (see Section 2.1, Shell Strain Displacement Equations). It was found that the non-HMN stress predictions in all cases showed large oscillation behavior both within and between elements, while the HMN stress results were very smooth and plausible in character over the entire structure. The membrane stress resultant parallel to the circular arc was found to be sensitive in both magnitude and distribution to whether the deep or shallow type of shell equations were used, for the case of the thick (R/t = 10) shell. The other stress resultants were not influenced by the choice of strain equations. Most of the stress resultants were strongly and plausibly affected by Poisson's ratio; this is discussed later in this section.

The extremely strong influence of the HMN freedoms and constraints can be understood quite clearly in this problem. It is recalled that the element displacement states, referred to the updated element baseplane, are cartesian. Considering an initially curved element, such as those employed in the present example, it is easily seen that a flattening of the element, due to bending displacements only, will cause appreciable compressive straining near its ends. The compressive strain due to such a flattening is given by $w^{\circ}{}_{,x} \cdot w_{,x}$, where $w^{\circ}$ is the initial curve of the element, w is the incremental elastic deformation, and x is the baseplane cartesian coordinate along the length of the element. The HMN displacements were specifically constructed to eliminate this type of straining, which has a second degree behavior in x, and, through the HMN constraint equations, this strain and the corresponding stress are eliminated. The important thing to note is that this effect, as described here, is purely linear. The high degree strain is due to linear behavior in the presence of initial element curvature, and results from the use of the cartesian baseplane coordinate system. The continued deformation of the element of course causes the development of nonlinear strains of the same type, which are also removed from the deformation state by the HMN functions and constraints. It might be argued that a change from the cartesian coordinates to the shell mid-surface curvilinear coordinates would eliminate the high polynomial degree strains in the linear case, this removing the need for the HMN functions and constraints. While this is true, of course, it would in turn cause much more serious calculation problems in the nonlinear case. This is discussed under "Shell Equations," and "Coordinate Systems and Updating," in Section 2.1.

Several plate bending problems, including the present one, have showed a nonlinear effect involving bending in the presence of a non-zero Poisson's ratio. Consider a plate or shell element initially flat in the y-direction, but either curved or flat in the x-direction. When the element is bent into an elastic curve in the x-direction, because of Poisson's ratio it attempts to respond by becoming concave in the y-direction on its surface of bending tensile stress. This action causes slope changes of the plate or shell, given by $w,_y$, which create y-direction nonlinear membrane tensile strains proportional to $(w,_y)^2$, particularly near the edges of the element. The HMN elements eliminate these strains through the HMN v-direction membrane displacement functions. The non-HMN elements cannot modify these strains, and the corresponding stresses, in a similar way, because the basic AZI-element membrane functions do not provide displacements of suitable polynomial degree. Hence, the non-HMN elements experience a strong, nonlinearly-induced resistance against the cross-curvature due to Poisson's ratio, and respond by severely limiting the amplitude of the y-direction curve of the elements. In the present problem, the HMN element under the load experiences a lateral curve of amplitude .0012", while the non-HMN element amplitude is .00028, for the case of the 67.9# load and the .025" thickness. Due to the lateral curve, the x-direction membrane stress for the HMN element is significantly affected due to the resulting geometry change. Near the load, the edges of the elements tend to move toward the center of the arc as a result of the Poisson's ratio-induced lateral curvature. This causes a rather large x-direction compression stress near the element edges, and a large tension stress near the central portion of the element. This behavior is undetectable for the non-HMN case, for two reasons: the lateral curvature is negligible; the x-direction membrane stress is very badly behaved because of the lack of the HMN function action to handle the initial curvature-induced stresses (see discussion above).

Reference 7 shows a strong dependence of numerical results for this problem, for $R/t = 10$ and $\nu = 0$, between finite element solutions based on deep shell and shallow shell equations. This effect was evaluated with the present element, also for $R/t = 10$ and $\nu = 0$. The results indicated that the deflections and all stress resultants except the circumferential membrane stress are only slightly affected by the inclusion or omission in the strain-displacement equations of the terms such as $w^\circ,_x \cdot u,_x$ and $w,_x \cdot u,_x$, in the circumferential bending curvature. Moreover, the bending displacements in both cases check closely with the exact solution (linear). In the present formulation, based on cartesian displacement definitions, including or omitting these types of terms is the only way in which shallow or deep shell types of theories can be simulated. It is concluded that, for the type of strain displacement equations used in this research, the usual errors associated with shallow theory do not occur to any noticeable extent. This appears to be due to the fact that, for the cartesian displacements used, the usual approximations of the shallow shell type no longer cause any significant approximation in the equations for the curvatures and the twist. It was noted that the circumferential stress resultant was quite strongly affected by the change of equations. This may or may not be significant, however, because this particular resultant is not well predicted for the coarse finite element model used, and is extremely sensitive to the shell equations used in a way which is dependent on element size (Reference 7).

Example 2: Cantilever Plate with Initial Curvature

Figure 11 shows the problem under consideration. This is the interesting "carpenters tape" problem, in which nonlinearity occurs because of flattening of the cross-section. Many related problems are of importance in structural analysis. This structure acts initially as a simple cantilever beam. For either up-loads or down-loads, the edges of the beam tend to deflect in such a way as to flatten the initial lateral curve of the cross-section, thereby reducing its effective bending moment of inertia and causing loss of stiffness. The reasons for solving the problem in the present work were: to determine whether the HMN functions would act (as they should) to facilitate the flattening of the section, which for the non-HMN case should be greatly attenuated; to study the HMN-element behavior for a case of initial curvature transverse to the principal loading direction. This problem illustrated several very interesting facets of the behavior of HMN elements.

Overall, the solutions obtained showed noticeable loss of stiffness, due to cross-sectional flattening, for the HMN-element cases, and a slight tendency for stiffening with the non-HMN element. The cross-section showed the expected flattening for the HMN-element cases and also for the non-HMN element for the up-load case only. For the down load cases, the non-HMN element showed a reverse-flattening, i.e., an increased "cupping" of the section. The nonlinear effects were small at the load levels studied, and the HMN and non-HMN results were comparable in overall stiffness. Flattening of the section reached about 10% of the initial curved shape, and end deflections reached about .36", which is about 22% of the span. This deflection is much greater than either the thickness or the initial out-of-flatness of the section.

The flattening behavior computed with the HMN elements was well behaved and completely plausible. The somewhat greater stiffness and reduced flattening for the up-load case is consistent with the easily observed behavior of a carpenter's tape subjected to upward and downward directions of loading. This behavior is also consistent with the general effect of cross-curvature induced by Poisson's ratio, and may possibly be related to this effect. The ability of the HMN-elements to flatten is due directly to the action of the HMN functions, since it is through these functions that the development of large, y-direction membrane strains, due to the flattening, is avoided.

The anomolous flattening action of the non-HMN elements is caused by a very interesting and unexpected behavior. Due to the overall bending loading in the case of an upward load, the cross-section experiences tensile stress at the edges and compressive stress at the crown. The reverse occurs for a downward load. Considering first the downward load, it is seen that Poisson's ratio will tend to create compressive y-direction stresses near the edges of the element and tensile stresses near the crown. This y-direction stress is roughly quadratic in the y-coordinate. At this point a significant characteristic of the AZI element, and of many multi-mode finite elements, is noted: the element has no facility at all, within linear theory, to rid itself of these particular Poisson's ratio-induced stresses, because of the character of its available displacement functions. In the present case, however, where nonlinear strains are included, the element can can rid itself of these stresses almost completely. The means of doing this is by "cupping" the section to an increased lateral curvature. This action produces, through the terms $w^{\circ}{}_{,y} \cdot w_{,y}$ in the y-direction membrane strain, a tensioning of the outer edge zone, and, in conjunction with the basic element v displacements, a compression in the crown region. This action tends to reduce the potential energy. Thus, for the case of downward loading, the section becomes more laterally curved, and the overall structural behavior becomes somewhat stiffer. Of course, the well known tendency of the section to flatten (because of equilibrium effects) is present in the non-HMN as well as the HMN-element. It simply occurs in the present case that the above-described behavior due to the Poisson's ratio effect dominates the behavior. The flattening tendency is driven by relatively small forces, and cannot compete with the very dominant effects associated with the membrane energy.

For the upward load case, the non-HMN element must flatten its section to conform to the Poisson's-ratio-related behavior as discussed above. In this case the Poisson's-ratio-induced behavior reinforces the natural tendency of the section to flatten. The data (Figure 11) show, however, that, while the section does flatten in this case, it does so only to about the same degree that the "cupping" occurred for the downward load case. This behavior appears surprising, but is easily explained. The explanation is again in the dominant effect of the membrane energy. The lateral bending of the section has been seen to create large y-direction membrane strains, and thus affects the membrane energy directly. For the non-HMN case, which cannot relieve these membrane stresses and strains, the equilibrium-driven flattening tendency (at small loads) is simply not strong enough to overcome the inherent, membrane strain-induced stiffness against lateral bending (flattening). Thus, the non-HMN element solution shows a flattening or cupping of the section which is totally dominated by, and essentially serves to exactly balance, the Poisson's ratio-induced y-direction stresses. As observed, the effect is essentially exactly reversed for the upload and download cases. For the HMN-element, however, the lateral bending is only resisted by the small plate bending stiffness, because the HMN functions eliminate any y-direction membrane stress participation. Thus, the HMN elements show section flattening in direct response to the normal flattening tendency, as driven by the equations of equilibrium in the deformed structural shape.

One final observation concerning the HMN-element results for this problem is noted. The Poisson's ratio-induced y-direction membrane stresses due to the initial bending stress distrubution, seen to dominate the non-HMN element behavior, gives rise to absolutely no response in the HMN-element. The element does not have any first-order ability within its basic functions to rid itself of these stresses, because of the simple forms of the available v displacement functions (nor does the non-HMN element). It cannot use the HMN functions either, because the HMN constraints, which are the only means of activating the HMN functions, respond to strain only, and not to stress. Finally, it cannot rid itself of these stresses through lateral curvature, as the non-HMN elements do, because the HMN constraints prevent the development of y-direction deformations in this case (of course, this particular behavior is very undesirable anyway). The conclusion which results from this discussion is that the HMN element conceptual basis should be extended to apply to high polynomial degree stresses as well as strains. In effect, this would require committing the entire HMN process to the control of the potential energy theorem.

Example 3: Cantilever Plate with End Restraint

Figure 12 shows the problem under consideration. This structure is highly nonlinear due to the effect of the displacement constraint at the right end. Its deflection magnitudes are controlled almost entirely by the large tension stresses induced by this constraint, which are distributed almost uniformly over the span. The HMN and non-HMN elements produce virtually identical results under these circumstances (about 1% difference). The problem was solved primarily as a test of the solution procedure. In order to obtain convergence in this case, it is necessary to avoid, or otherwise attenuate the effect of, the extremely large residual loads which result from the initial steps of the solution, during which the stiffening effect due to the end condition is not present in the stiffness matrix. The present solution procedure (Section 2.3) handles this by looking at a range of amplitudes of the initial solution step displacements. A state of minimum residual is sought within this range to determine the optimal amplitude. In the present case, a minimum does not occur within the search range, and its lower limit amplitude is selected. The second step proceeds similarly. Because of the rapid change of stiffness with deflection, the stiffness matrix is updated frequently. This type of solution procedure avoids the requirement that the user set intelligently chosen amplitude factors for the attenuation of the residual load iterative procedure. It also avoids divergence even with relatively large load steps. Prior to implementation of the optimum-amplitude search-type procedure, solution attempts diverged for this problem.

The figure shows a plot of the deflected shape of the edge of the plate. The centerline deflections are slightly different due to the .3 value for Poisson's ratio. The initial linear steps had an end deflection of .155" for zero Poisson's ratio and .148 for $\nu = .3$. The linear theory end deflection for zero Poisson's ratio is .157". The straight-line character of the deflection shape for the outer two elements is a result of the nonlinearity-induced tensile stresses. The nonlinear solution gives about 20% of the linear solution value of the end deflection, at maximum load.

Example 4: Twisting of a Square Plate Fixed on One Side

Figure 13 illustrates the problem, which was solved with both a single element and with four elements, as indicated. For the four element model the applied loads are distributed over the end such that the deflections there are linear in y. The applied torque is 250 inch-pounds. The character of this problem is dominated by the use of the total fixity condition at the supported edge. The problem is significantly nonlinear, with the initial step, linear solution maximum deflections exceeding the final converged deflections in the ratios of 1.65 for the single element and 1.87 for the four element case. The membrane stresses are (and should be) quite large for this problem, and are important influences on the torsion-bending behavior. Because of the fixed edge and the fact that the structure would "prefer" to adopt a deformation state of constant, pure twist, rather than the torsion-bending state demanded by the edge constraint, the single element model is much too stiff in the linear case. The linear first step single element maximum displacement is .161", while for the four element model the value is .224". The deflection data are tabulated on the figure.

For the case of the fixed edge, this problem is one in which the HMN elements as formulated in this research demonstrate unacceptable behavior. The difficulty is in the x-direction variation of the y-direction direct strain, $\varepsilon y$. This particular behavior ($\varepsilon y$ vs. x) is not considered by the HMN constraints or the HMN supplementary displacement functions. The slope $w,_y$ is a second degree function of x because of the fixed edge. Thus, the nonlinear behavior generates a fourth degree function of x in the strain $\varepsilon y$. No displacement options are available to counteract this strain, which remains in the deformation state as a source of excessive energy and, hence, excessive stiffness. Solutions of this problem with both the single and four element models show that the stress $\sigma_y$ varies as a high degree (4th) function of x. Since this occurs for both the $\nu = 0$ and $\nu = .3$ cases, it is clearly caused by slope $w,_y$ and the inability of the present HMN formulation to deal with the problem in a successful manner. The largest $\sigma_y$ values occur at the loaded end of the plate, of course, because the lateral slope is largest there.

The figure shows a "carpet" plot of the variation of $\sigma_y$ vs y on five x=constant lines on the plate. Both the single element and four element cases are shown. For the single element model, for which $\nu = 0$, $\sigma_y$ is constant in y, and the implied cross-plots would show $\sigma_y$ to be quartic in x. For the four element model, for which $\nu = .3$, the effect of Poisson's ratio is to cause $\sigma_y$ to be quadratic in y over each individual element. The distribution is symmetric about the x-centerline, but a small discontinuity in $\sigma_y$ occurs between the elements which join each other at $x = .5$. Again, the implied cross-plot variation of $\sigma_y$ *with* x *is quartic*. The four element model shows comparable $\sigma_y$ values to the single element model overall, but has considerably larger values at the loaded end. The cause of these larger stresses lies partly in the added twist angle of the four element model (about 23%), and partly in the rather sharply peaked distribution of $\sigma_y$ with y. The latter is probably due to the effect of Poisson's ratio.

The figure indicates the regions of tension and compression $\sigma_x$. In both models this stress is compressive over the central region ($0<X<1$; $.25<y<.75$) and tensile over the edge regions ($0<X<1$; $o<y<.25$; $.75<y<1$). The cause of this behavior is of course the tendency of the edge zones to foreshorten, due to their slope, which tendency is resisted by the central zone of the plate. This gives rise to a rather large level of membrane shear stress as well. The single element and four element models have roughly comparable maximum values of the x-direction direct stress resultant and the shear stress resultant, as follows: four element model, $N_x = 21300$, $N_x = -8000$,

$N_{xy}$ = 5800; single element model, $N_x$ = 7300, $N_x$ = -5000, $N_{xy}$ = 8300. The larger values of $N_x$ for the finer model reflect its ability to adequately represent the x-direction distribution of the stress; the $N_x$ versus x plot shows a maximum at x = .5.

The HMN functions have the potential to cause inter-element incompatibility of the x-direction displacement between the joining elements at x = .5. The data were checked for the finer model, and this incompatibility was found to be present, though small. This results from the strong twisting action of this problem, resulting in large, nonlinear membrane shear stresses. This is considered to be an unsatisfactory behavior on the part of the HMN elements in their present formulation.

The conclusions reached from this analysis of nonlinear torsion-bending are: the HMN formulation must deal with the nonlinear strains $\varepsilon_x$, $\varepsilon_y$, and $\varepsilon_{xy}$ as a coupled set, no one of which can be optimized in any way independent of the others; the possibility of inter-element incompatibility should be eliminated by controlling the HMN displacements by nodal freedoms. These conclusions point toward a formulation in which the HMN theory is based on potential energy constraints only, and manipulates displacements which are entirely nodally controlled.

<u>Example 5</u>: BCIZ-HMN Cantilever Beam

Figure 14 shows a three element cantilever beam bent by end loads. The initial problem solutions with this model used z-direction concentrated loadings at nodes 4 and 5. The results for this loading showed the beam to be about 20% too stiff, and also yielded a free end moment of 45% of the fixed ended moment, within the outer pair of elements. The net end moment at the free end was zero, of course, as the result of the combined action of all three elements. The residual x-direction stresses along the outer edges of the element were proportional to the square of the load (hence related to nonlinear behavior), about 1% of the potential nonlinear stress, and of quadratic variation along the edge, rather than the expected linear variation. These results indicate that though the HMN procedure is functioning correctly, some unexpected nonlinear effect is preventing the basic BCIZ membrane functions from eliminating the nonlinear x-direction stresses. It was concluded that, probably due to the BCIZ element's characteristic of permitting inter-element slope incompatibilities, the minimum energy state of the element actually retains small membrane stress levels. More important, the 20% stiffness error and the erroneous bending moment distribution were judged unacceptable, even for the coarse idealization employed, because these elements use cubic displacement functions, and should have been very accurate for this problem. Therefore, this problem was investigated further under different loadings, as indicated in Figure 14. The loadings are, respectively, concentrated nodal moments, generalized loadings based on a distributed edge moment along edge 4-5, and loadings deduced directly from the element stiffness matrices to obtain a constant curvature state. The numerical results obtained, as given in the figures, show the model to be accurate within 1% for the nodal moment case, to be about 20% soft for the generalized (work-equivalent) loading, and to be about 6% soft for the loading derived from the stiffness matrices. These percentages are based on "proportional" loading, and refer to the magnitudes of the moment loadings themselves. The accuracy obtained for the pure nodal moment case is excellent, but of questionable meaning. That obtained in the generalized load case is clearly unacceptable. The 6% error for the constant curvature case is especially surprising, since the elements should have responded to this deformation in an exact way. It is noted that the residual stresses are identically zero for this case. In the case of a structure incorporating many of these elements, it is difficult to say what sort of accuracy would be obtained. For constant curvature, the BCIZ elements should not exhibit inter-element slope incompatibilities. It is concluded that further investigation along these lines would be required, before the value of the BCIZ element as a basis for a stability element could be established. Because of the strong coupling between the HMN freedoms and the slopes, the lack of slope continuity inherent with the BCIZ element may be a serious difficulty. These difficulties caused the BCIZ-HMN element to be dropped from the present research, in favor of the AZI-HMN element.

## 2.6 Alternatives to the HMN-Element Approach

This section attempts to discuss alternative methods which may provide equivalent gains to those obtained with the HMN approach, in application to strongly nonlinear problems. The discussion derives from basic reasoning concerning the goals and difficulties of solving nonlinear problems, and not from direct comparative numerical evaluations, except for the specific AZI-element based HMN versus non-HMN comparisons of the previous section. Thus, no claim of completeness is made here. In addition, the matter of HMN-element performance of initially curved elements in linear analysis, which was discussed in Sections 2.1 and 2.5, is not specifically addressed again here. However, most of the conclusions drawn herein for nonlinear analysis are also applicable to the linear analysis of initially curved elements, for element types whose formulations are specifically tailored for nonlinear problems.

The accurate and cost-effective finite element solution of nonlinear problems would appear to depend on such factors as the following:

1) ease of discretization and reasonable control of problem size (degrees of freedom)--the use of reasonably large and reasonably uniform sizes of elements

2) avoidance of excessive computer run times and costs, as regards element-related calculations (stiffness matrices, decompositions, residual loads)

3) reliable and reasonably fast convergence of residual load iterations--the ability to use relatively large load steps

4) avoidance of the likelihood of an aborted solution or a solution which has converged to an incorrect result

The HMN procedure was aimed principally at item 1); i.e., it is designed to solve linear and nonlinear problems with equal accuracy for the same element sizes. At this point it cannot be concluded whether it is particularly advantageous as regards items 2) and 3). It does well with respect to item 4), converging to correct results in cases where non-HMN elements of the same size and basic formulation do not.

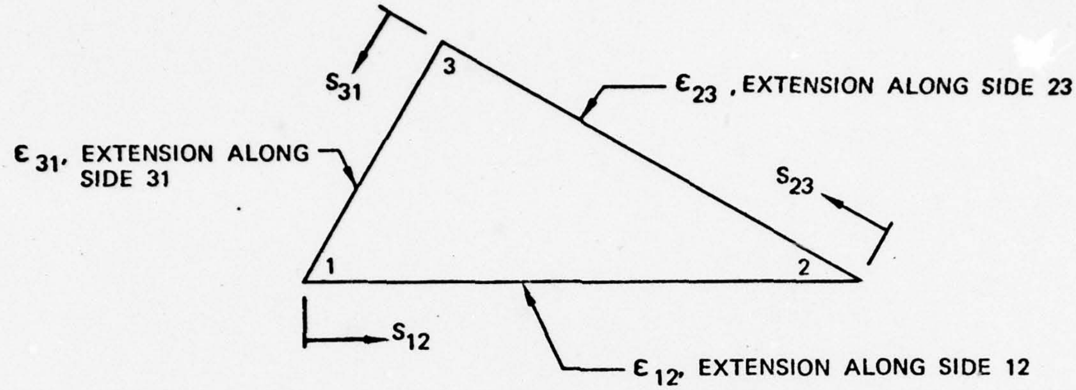
It appears that all of the technical gains of the HMN elements can be obtained with conventional elements by simply using smaller element sizes in a fully nonlinear solution. This is a very attractive alternative (although admittedly most currently available conventional elements do not retain full nonlinearity of strains), and one which probably would not increase computer costs excessively for most problems. For very large problems, however, the rapidly increasing cost of decomposition of the structural stiffness matrix, as the number of elements is increased, would become a serious problem. Perhaps the most serious objection to this approach is that the analyst's task is made considerably more difficult in this case. The necessary element sizes are not easy to guess before the first numerical solutions are obtained, and it will be true in many problems that markedly different sizes of elements will be required in different areas of a structure, due solely to the local degree of nonlinearity of behavior. For the HMN elements, on the other hand, the analyst is free to use the same discretization which would be suitable for linear problems. In addition, once a solution has been obtained with small, fully nonlinear, conventional elements, the question of whether a significant nonlinearity-induced error has occurred would have to be addressed. A direct approach to answering this question would be to compare the membrane strain magnitudes caused by the linear and the nonlinear deformations. This is not always a satisfactory approach, however. The difficulty is that the conventional elements are constrained (by the global equations and the residual loads) such that they underestimate the bending curvatures, particularly in zones of significant nonlinearity. Thus, the true magnitudes of the nonlinear strains may be considerably larger than the apparent values given by the solution obtained. Based on these considerations, it would appear that, overall, the HMN type of element has a significant advantage over the use of smaller conventional elements, in most problems.

Another possible approach involves limiting the types of terms which are included in the calculation of nonlinear strains. For example, in the case of an element which retains linearly varying membrane displacements, the computation of nonlinear strains might consider only the average bending slopes of the element in the calculation of the direct strains. A comparable assumption would be made for the membrane shear strain. By this means the nonlinear strains are kept to the same functional forms as the linear strains, and hence excessively stiff behavior is not caused specifically by higher degree strain polynomials due to the nonlinearity. This particular approach has a technical drawback, however: the elements will not respond to the tendency of tensile stresses to straighten an element, or of curvature increases to shorten an element (or the reverse for compression, curvature decrease). This drawback limits these elements to use in small element sizes for nonlinear problems. Since the type of error involved here acts directly on the high energy overall tension/compression/shear membrane stress state of the element, rather than on only a higher degree polynomial component of these stresses, it would appear to have the potential for causing serious errors in equilibrium. It is noted that taking this type of liberty with the strain equations necessarily affects the coupling between nonlinear membrane direct strain and shear strain. If this approach were used for an element such as the AZI element, in which the membrane displacements have a second degree polynomial capability, and hence a first degree membrane stress capability (second degree for shear), there is not available an efficient truncation of the bending slope polynomial. Use of the average slope only would appear to be an overly severe approximation, but retention of the linear term in the slope already exceeds the basic element membrane strain polynomial capability. Thus, this basic approach to handling nonlinear behavior would appear to be most efficient in application to the simplest type of element as regards membrane displacement--that in which the membrane displacements are linear functions. Such elements, however, are usually not accurate for either linear or nonlinear analysis of curved shell elements except in very small element sizes. This is because, for curved shells, generally membrane displacement behavior is as rapidly varying as bending behavior, except in local edge-bending zones. In addition, without the use of competent membrane functions, there is often difficulty in providing finite elements with strain-free rigid body motions, especially for large deflection analysis.

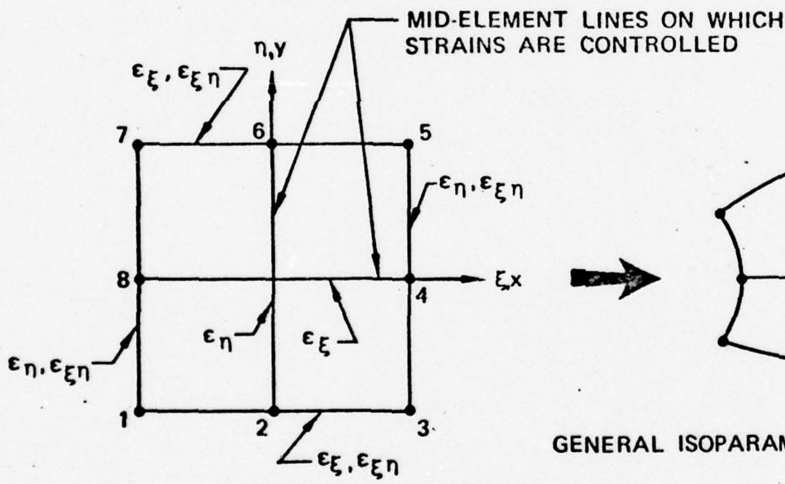
A similar approach to the above would be one in which the fully nonlinear strain is computed, but the result is filtered in some way, retaining only lower (basic element) degree polynomials for stiffness matrix and residual load calculation. We are not aware of such an approach being used, and it is difficult to guess what sort of performance it would provide. It would appear that the method has a basic flaw in that the membrane strain state and the bending slope state do not properly correspond to each other. Thus, bending displacements might be incorrectly predicted, because their membrane (high energy) consequences are not fully accounted for. This is much the same sort.of objection as the arguments given above against the use of average slopes only in the nonlinear strain equations.

It would be possible to use a very complex element as regards all types of freedoms. Thus, for example, an isoparametric element with 5 nodes per side might be used. This would provide adequate membrane strain capability to handle nonlinear strains caused by bending slopes corresponding to a constant curvature state. Of course, the possibility of very high degree nonlinear strains exists in this case, because of the high degree bending displacements employed. The constant curvature terms are dominant, however. It would be possible to use relatively large elements of this type in nonlinear problem solutions, with results comparable to those obtained with the HMN type of element. This appears to be a superior approach, and is similar to the recommended approach arrived at on the basis of the present research.

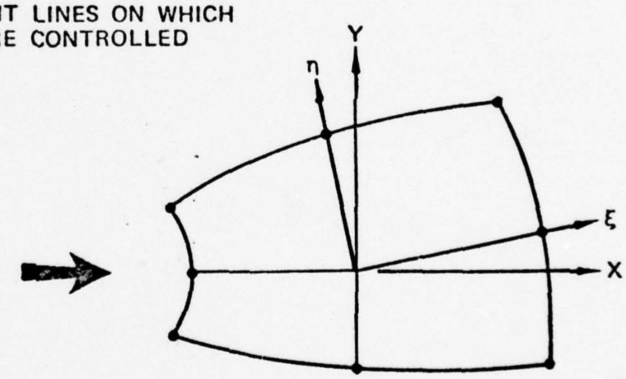
All of the above possibilities relate to problem solutions in which nonlinear strains are computed, and the resulting nonlinear stresses affect the residual loads and stiffness matrices of the elements. There are also methods which perform stepwise solutions in which nonlinear effects are totally ignored, so far as explicit calculation is concerned. These methods always use a "geometric" stiffness matrix to represent the effort of structural shape change on the equilibrium equations. This is probably the most common type of method employed for solving nonlinear problems in current finite element programs. The method commonly takes two forms: in one case, no iteration is done, and the structural geometry is updated, forming elements of new shapes, at the end of each step; in the other case a single step is done and iterated, with the iteration based not on residual loads but on re-forming the "geometric" stiffness matrix to reflect end-of-step values of the stresses. In the first case, nonlinearity is handled implicitly by the updating of the shapes of the elements. However, the nonlinearity which occurs during a step is not included because residuals are not evaluated. This approach is known to yield problem solutions which deviate increasingly and significantly from correct solutions, even if very small steps are used. In the second method, no nonlinearity is considered at all. At the completion of a single stepwise calculation, the end-of-step stress state is used to evaluate an updated stiffness matrix, and the stepwise calculation is repeated. The end-of-step stresses are computed using linear theory. This method obtains a correct solution only if nonlinear strain effects are negligible. It would be possible to combine the methods, obtaining a procedure which should deviate more slowly from the exact solution that the above-described conventional stepwise method. In developing such a method for nonlinear problems, it would be necessary to deal carefully with the interpretation of the relationship between element stresses and forces, due to changing element shapes. This would prove somewhat complex, and it would appear that recourse to a fully nonlinear approach might be nearly as easy to develop. Nevertheless, this type of quasi-nonlinear analysis may have merit in some types of problems.




TRIANGULAR ELEMENT




PARENT QUADRILATERAL ELEMENT




GENERAL ISOPARAMETRIC QUADRILATERAL ELEMENT

FIGURE 1: STRAIN CONSTRAINTS FOR EXPLICIT CONTROL OF STRAIN POLYNOMINALS

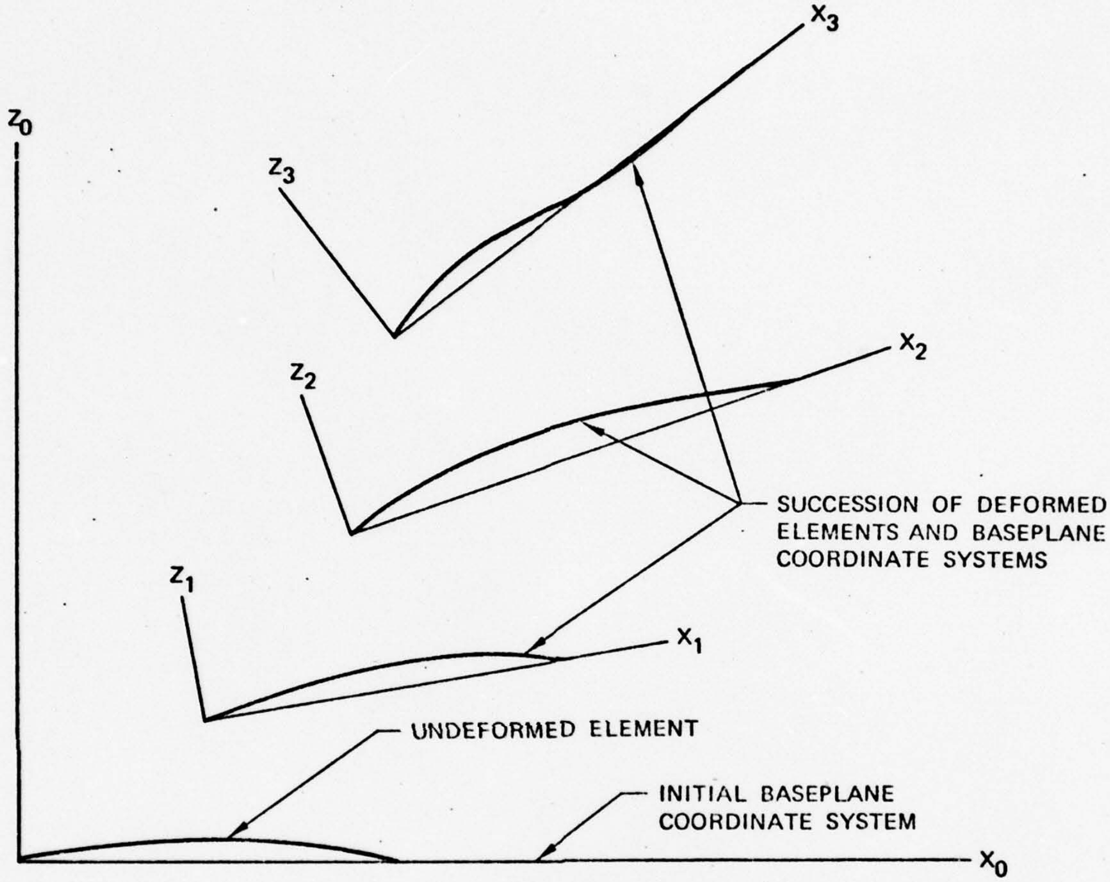



FIGURE 2: COORDINATE SYSTEM UPDATING – TRANSFORMED BASEPLANE COORDINATE SYSTEMS

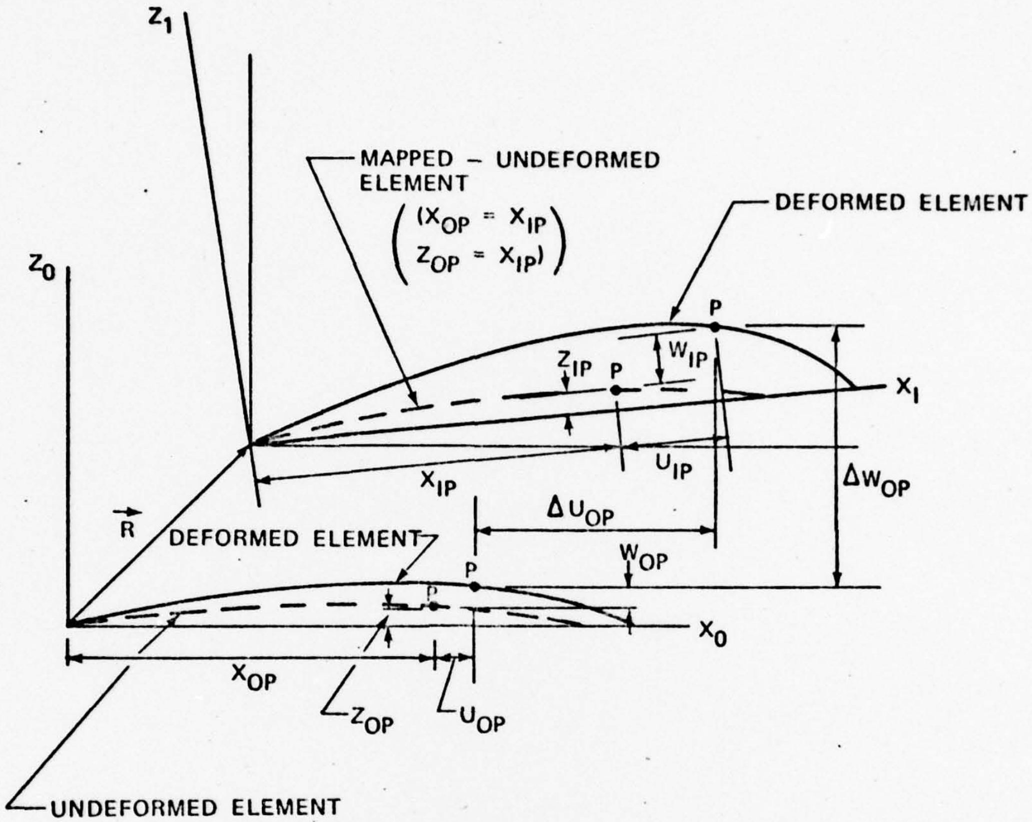



FIGURE 3: DISPLACEMENT TRANSFORMATION FOR BASEPLANE UPDATING

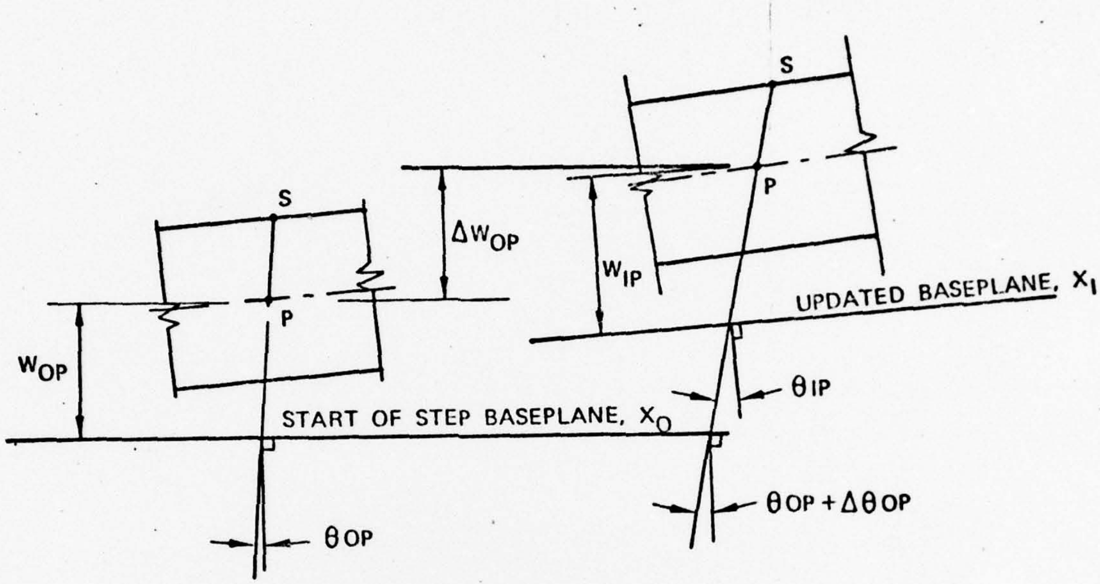



FIGURE 4: ROTATION TRANSFORMATION FOR BASEPLANE UPDATING

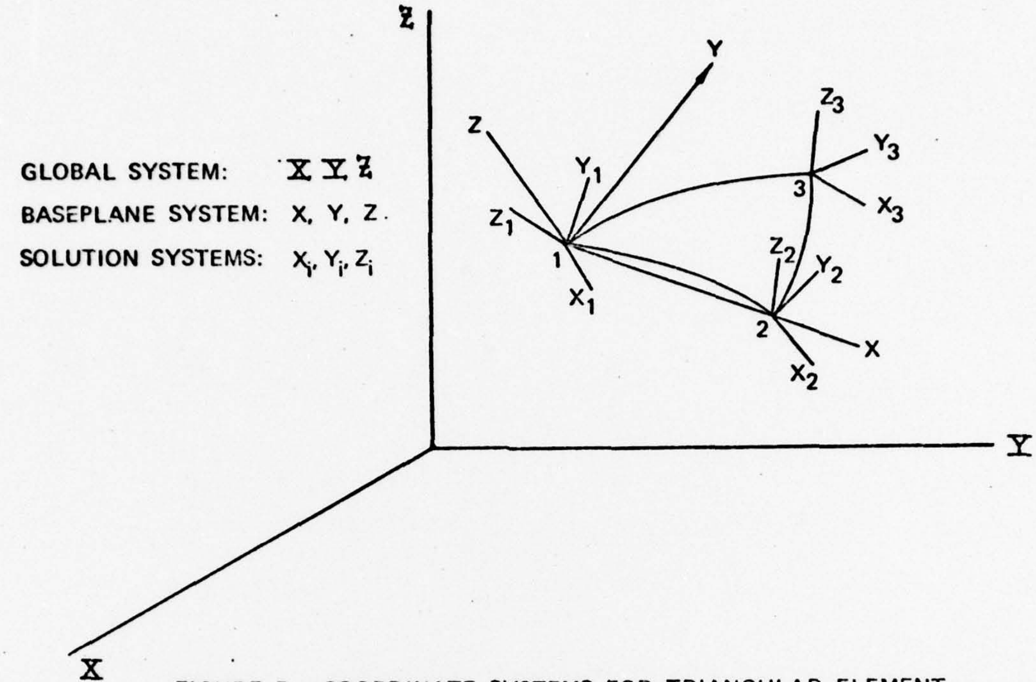



FIGURE 5: COORDINATE SYSTEMS FOR TRIANGULAR ELEMENT

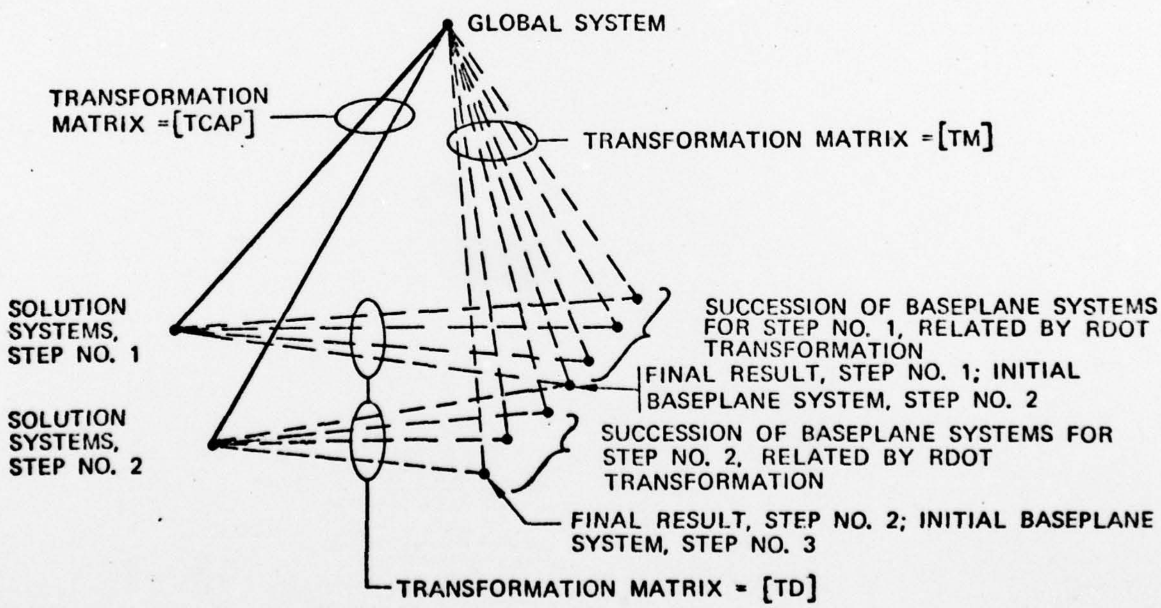



FIGURE 6: COORDINATE SYSTEM SUCCESSION FOR STEPWISE SOLUTIONS

$$K_{51 \times 51} \xrightarrow{\text{HMN}} K_{45 \times 45} \xrightarrow{\text{DELETE}} K_{39 \times 39} \xrightarrow{\text{MPE}} K_{27 \times 27} \xrightarrow{\text{"TSTAR"}} K_{27 \times 27} \xrightarrow{\text{TD}} K_{27 \times 27} \xrightarrow{\text{MERGE}} \text{MERGE STIFFNESS MATRIX}$$

TRIANGULAR ELEMENT TRANSFORMATIONS

$$K_{58 \times 58} \xrightarrow{\text{HMN}} K_{48 \times 48} \xrightarrow{\text{DELETE}} K_{44 \times 44} \xrightarrow{\text{MPE}} K_{40 \times 40} \xrightarrow{\text{TD}} K_{40 \times 40} \xrightarrow{\text{MERGE}} \text{MERGE STIFFNESS MATRIX}$$

QUADRILATERAL ELEMENT TRANSFORMATIONS

FIGURE 7: SEQUENCE OF STIFFNESS MATRIX TRANSFORMATIONS FOR STABILITY ELEMENTS

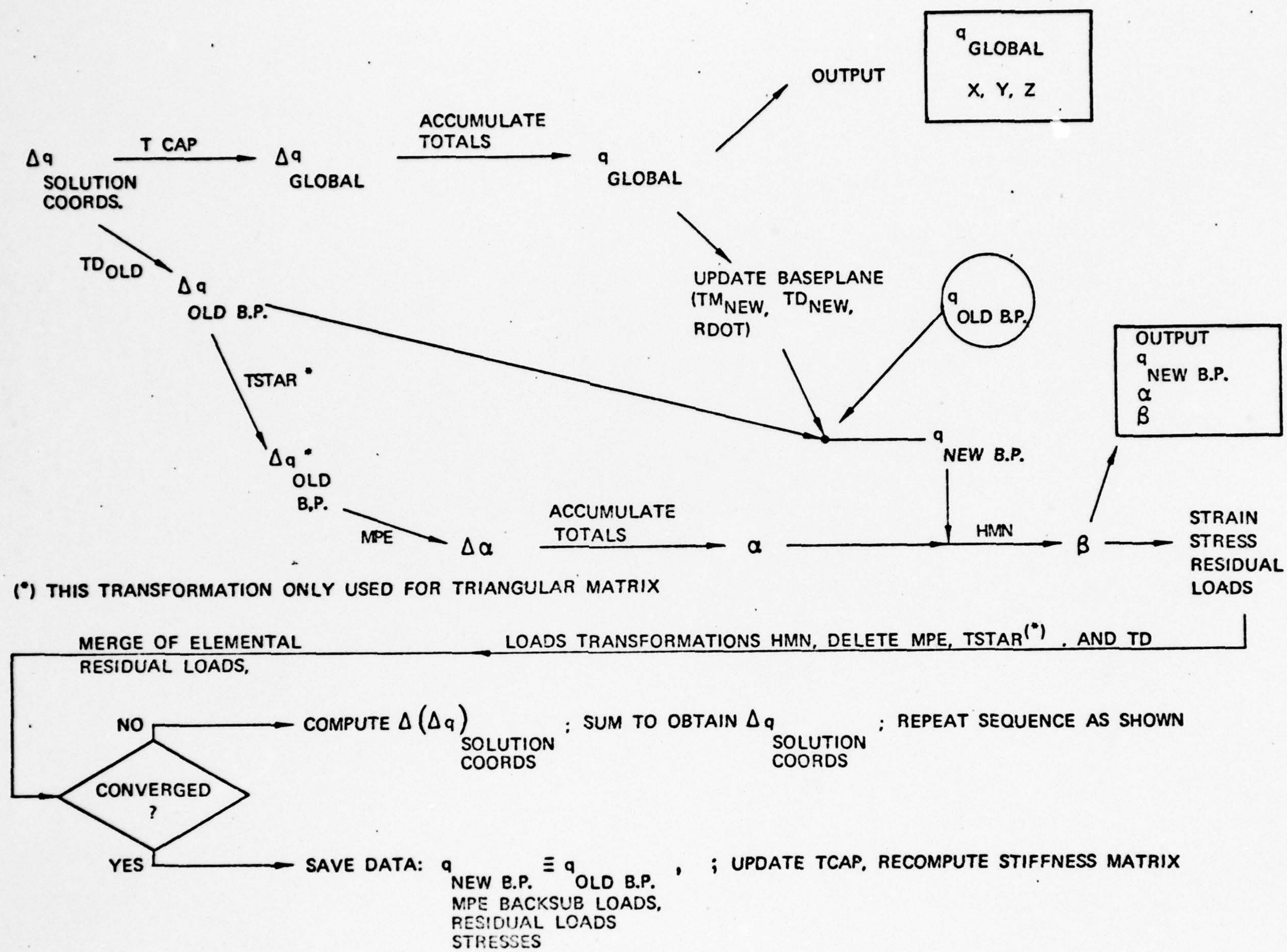
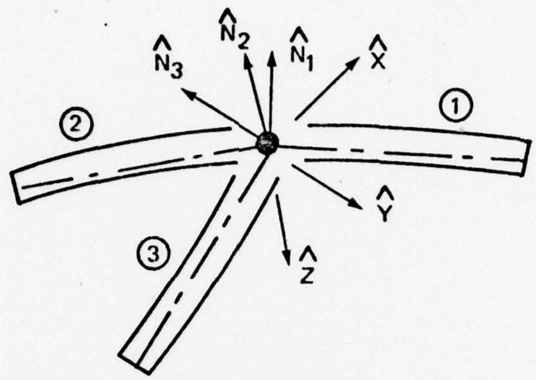
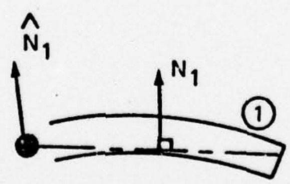



FIGURE 8: SOLUTION PROCEDURE AND DATA PROCESSING




a) NODAL ENTITIES – UNDEFORMED CONFIGURATION




b) DEFORMED ELEMENT

FIGURE 9: ROTATIONAL TRANSFORMATION

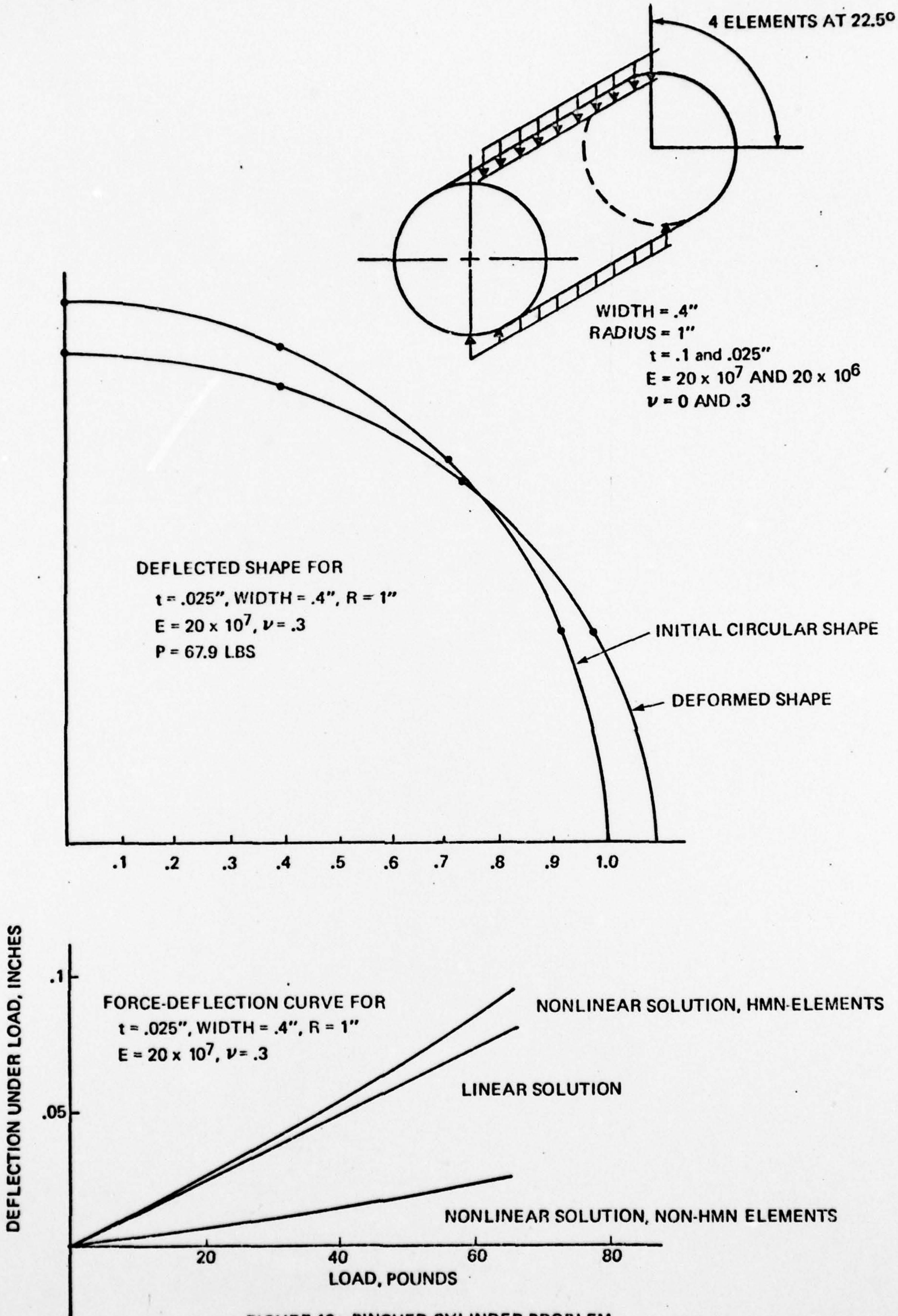



FIGURE 10: PINCHED CYLINDER PROBLEM

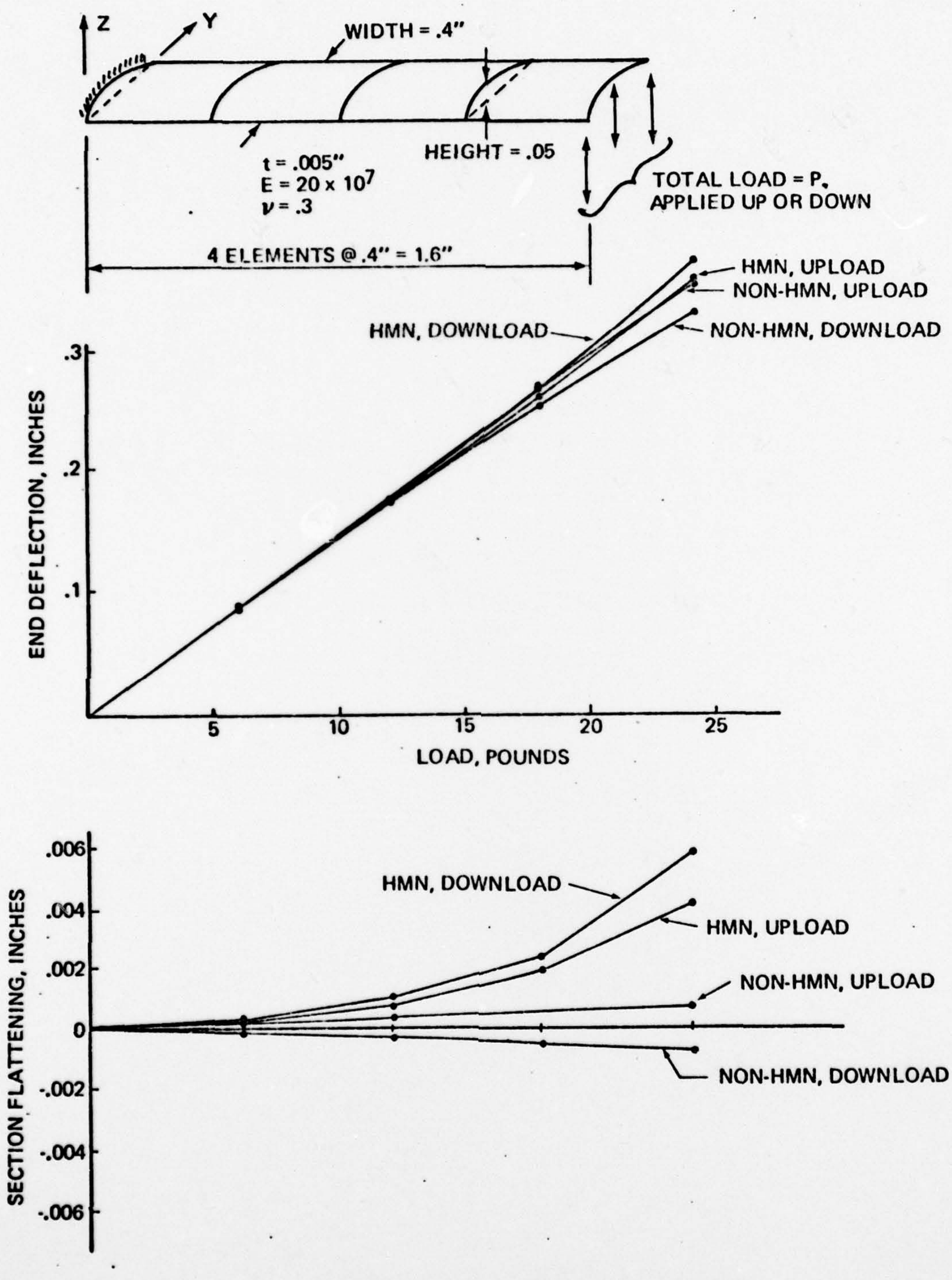



FIGURE 11: CARPENTER 'S TAPE PROBLEM

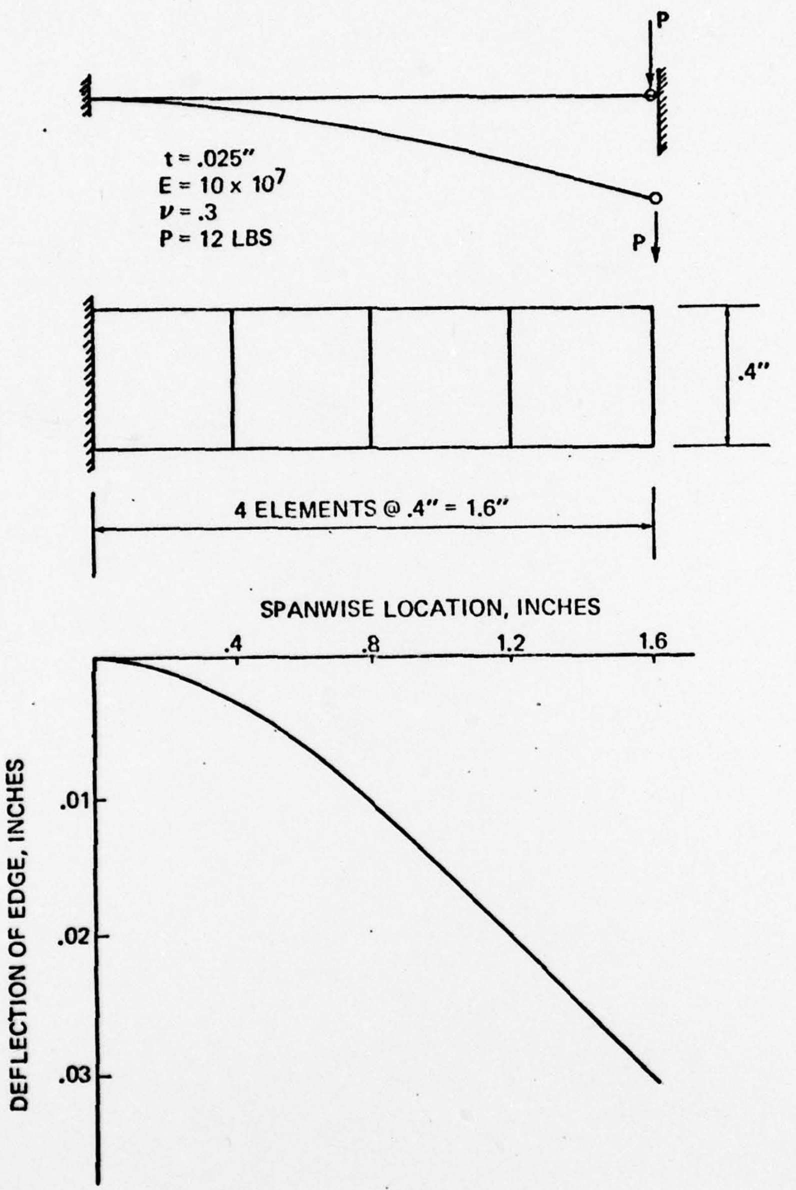
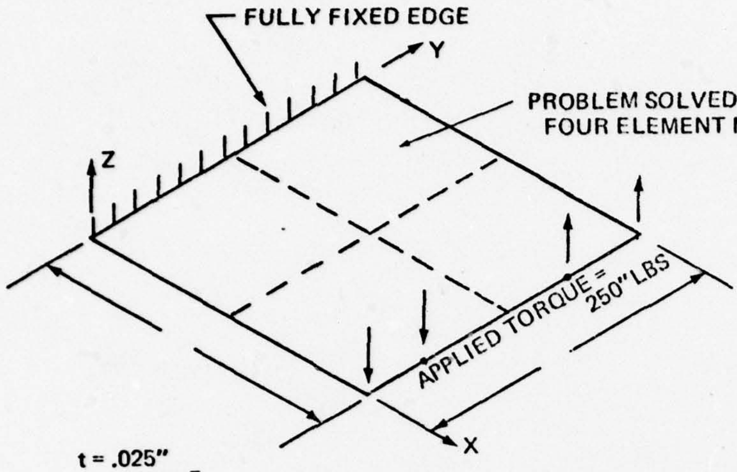



FIGURE 12: CANTILEVER PLATE WITH END RESTRAINT




| MAXIMUM DEFLECTIONS (IN.) | | | |
|---|---|---|---|
| SINGLE ELEMENT | | FOUR ELEMENTS | |
| LINEAR | NONLINEAR | LINEAR | NONLINEAR |
| HMN .161 | HMN .098 | HMN .224 | HMN .119 |
| NON-HMN .161 | NON-HMN .066 | NON-HMN .224 | NON-HMN .115 |

t = .025"
E = 10 x $10^7$
$\nu$ = 0.0 FOR SINGLE ELEMENT MODEL
$\nu$ = 0.3 FOR FOUR ELEMENT MODEL

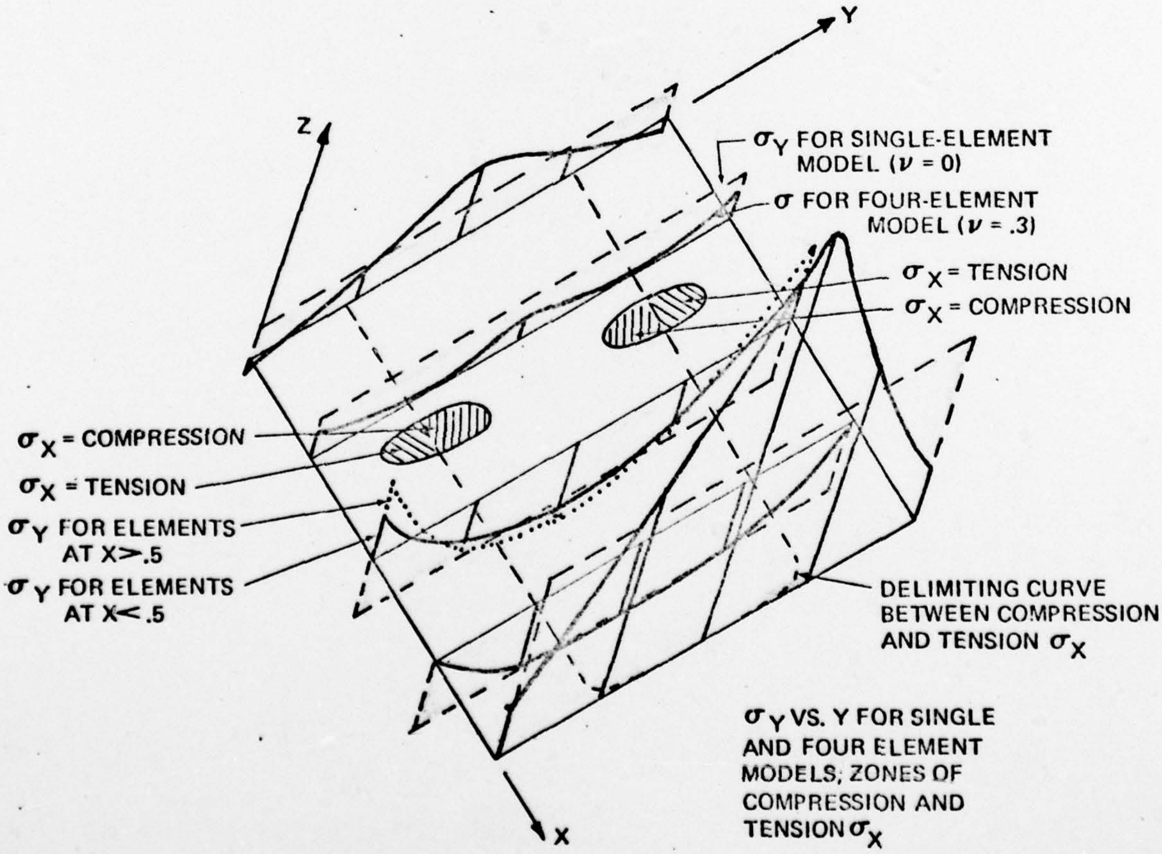
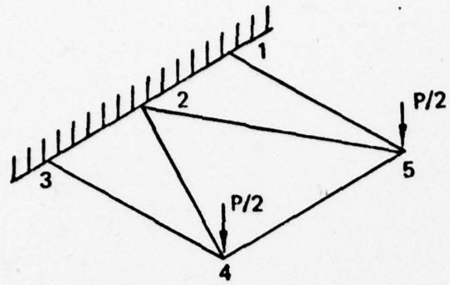
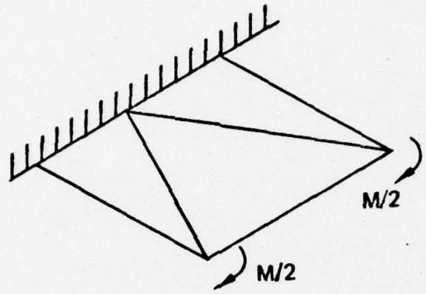


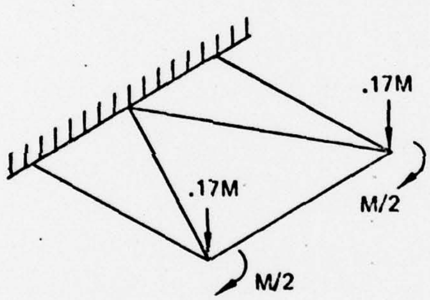

FIGURE 13: TORSION OF QUADRILATERAL PLATE



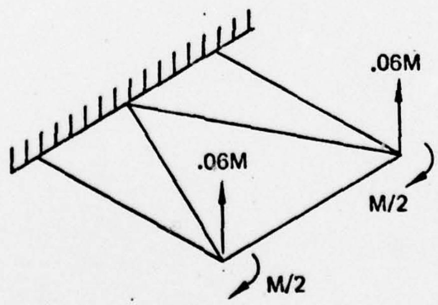

END LOADED CANTILEVER:
20% TOO STIFF;




CONCENTRATED NODAL MOMENTS: DEFLECTION AND ROTATION WITHIN 1% OF THEORY; NONLINEAR STRESSES VERY SMALL




GENERALIZED LOADINGS FOR LINE MOMENT ON EDGE 4-5: DEFLECTION AND ROTATION 20% TOO LARGE; NONLINEAR STRESS TOO LARGE




CONSTANT CURVATURE, WITH LOADINGS DETERMINED FROM ELEMENT STIFFNESS MATRICES: DEFLECTION AND SLOPE 6% TOO LARGE: NONLINEAR STRESSES $\approx$ ZERO

FIGURE 14 : ACCURACY AND LOADING OPTIONS FOR BCIZ-HMN ELEMENT

## 3.0 SUMMARY AND CONCLUSIONS

This report concludes a study of finite element analysis of plate and shell problems with strong geometric nonlinearities. The specific area of study is the influence of the nonlinear strains, which generally are of complex distributions over the elements, on element stiffnesses and accuracy of problem solutions. The means of investigation was through newly developed special types of finite elements, in which the conventional displacement functions are supplemented by added membrane displacements of higher order forms. The added functions are used to modify the nonlinear membrane strains in order to obtain the simple strain distributions characteristic of linear analysis. A combination of constraints is used to accomplish this. They include explicit constraints which set to zero the higher order polynomial functions in the nonlinear strains. Conventional potential energy-based constraints are also used. All constraints are done at the elemental level.

In order to implement the elements, solution procedures were developed, based on a Lagrangian formulation, in which element-following, updated coordinate systems are used to allow total strain to be computed without incrementation. The solution procedure uses linearized stepping plus iteration, with added features which account for in-step nonlinear behavior.

Evaluation of the special nonlinear elements was made through comparative solutions obtained with these elements and with conventional elements of the same basic formulation, all using the same solution procedure. This work has led to the conclusions outlined in the following paragraphs.

For strongly nonlinear problems, and also for linear problems with initially curved elements, the special elements perform much better than conventional elements when relatively large element sizes are used. The error characteristic of the conventional elements is excessive stiffness, amounting to a factor of two to four in some of the problems solved. The conventional and special elements become essentially equivalent in performance for sufficiently small element sizes.

For nonlinear analysis in which a fully nonlinear strain representation is included, the use of linearized steps generally yields displacement increments of incorrect "shape," in the sense that the relative membrane and bending displacements are incorrectly proportioned. In addition, for many problems, the error in displacement "shape" and the use of stepwise linearity causes the displacement increments to be of incorrect amplitude overall. These effects result in excessive residual loads and difficulty in convergence of the iteration process. This research has demonstrated the beneficial effects of including in-step nonlinearity by means of calculations made with a special solution procedure which approximates the in-step nonlinearity. It is recommended that a stepwise nonlinear approach such as the static perturbation procedure be used with the special types of nonlinear elements under study in this research.

The use of a stepping procedure which combines updating of element baseplane coordinate systems, total Lagrangian strain calculation, and locally cartesian displacement reference systems is particularly convenient and accurate for nonlinear analysis of shells.

The special nonlinear element formulation used in this research requires modification in order to achieve general applicability. The explicit constraint of higher order strain polynomials is not a satisfactory approach, because it interferes with the proper coupling between the three membrane strains. It is recommended that the formulation be modified, for isoparametric elements, as follows: (1) the supplementary high order membrane displacement functions should be controlled by nodal freedoms, employing additional nodes for this purpose; (2) the added nodes need not, and probably should not, be used for added bending freedoms; (3) all constraints relative to the supplementary functions should be enforced globally, based on conventional potential energy principles. A formulation of this type would be simpler to develop and implement than the one used in the present research, and would provide all of the benefits demonstrated for the special nonlinear elements herein.

## 4.0 REFERENCES

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